THE BALL BANACH FRACTIONAL SOBOLEV INEQUALITY AND ITS APPLICATIONS

YIQUN CHEN, LOUKAS GRAFAKOS, DACHUN YANG*, AND WEN YUAN

Abstract. The authors obtain a fractional Sobolev inequality for Sobolev spaces $\dot{W}^{s,X}(\mathbb{R}^n)$ for ball Banach function spaces X on $\mathbb{R}^n$ with the homogeneity and the non-collapse properties. Precisely, the authors show the existence of a positive constant C such that, for any $f \in \dot{W}^{s,X}(\mathbb{R}^n) \cap X^{\frac{\alpha}{\alpha+s}}$,

$$\|f\|_{\dot{W}^{s,X}(\mathbb{R}^n)} \geq C \, \|f\|_{X^{\frac{\alpha}{\alpha+s}}},$$

where α is the homogeneity index of X, $s \in (0, \min\{-\alpha, 1\})$, and $X^{\frac{\alpha}{\alpha+s}}$ is the $\frac{\alpha}{\alpha+s}$-convexification of X. Moreover, under some mild assumptions, the authors prove that the closure of $C_{\rm c}^\infty(\mathbb{R}^n)$ with respect to $\|\cdot\|_{\dot{W}^{s,X}(\mathbb{R}^n)}$ modulo constants is identified with $\dot{W}^{s,X}(\mathbb{R}^n) \cap X^{\frac{\alpha}{\alpha+s}}$. When X is a Lebesgue space, these results reduce to the well-known Sobolev embeddings for which the restriction $s \in (0, \min\{-\alpha, 1\})$ is sharp. However, these results also provide new Sobolev embeddings for Morrey spaces, mixed-norm Lebesgue spaces, Lebesgue spaces with power weights, Besov–Triebel–Lizorkin–Bourgain–Morrey spaces, and Lorentz spaces. As in the case for the classical Sobolev inequality, these results have a wide range of applications.

1. Introduction

It is well known that, for any given $s \in (0, 1)$ and $p \in [1, \infty)$, the *homogeneous fractional Sobolev space* $\dot{W}^{s,p}$ is defined as the space of all measurable functions f on $\mathbb{R}^n$ whose Gagliardo semi-norm

$$\|f\|_{\dot{W}^{s,p}} := \left[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}}\, dx\, dy\right]^{\frac{1}{p}} \tag{1.1}$$

is finite. Here and thereafter, since all function spaces appearing in this article are defined on $\mathbb{R}^n$, to simplify the presentation, we will not indicate their underlying spaces. The classical Sobolev embedding, also known as the fractional Sobolev inequality, states that when $sp < n$ one has

$$\|f\|_{L^{p_s^*}} \leq C \, \|f\|_{\dot{W}^{s,p}} \tag{1.2}$$

for any $f \in C_{\rm c}^\infty$ with the positive constant C independent of f, where $p_s^* := \frac{np}{n-sp}$ denotes the critical Sobolev exponent and $C_{\rm c}^\infty$ denotes the set of all infinite differentiable functions on $\mathbb{R}^n$ with compact support. We refer to [54, Theorem 10.2.1] for an elementary proof of (1.2) (see also [65, Théorème 8.1]). It is well known

2020 *Mathematics Subject Classification.* Primary 46E35; Secondary 26D10, 42B25, 26A33.
Key words and phrases. ball Banach function space, fractional Sobolev inequality, closure
* Corresponding author.

that the Sobolev type inequalities on various function spaces have received a lot of attention and intensive studies for a long time; see, for example, Haroske et al. [26, 28, 32], Nakai et al. [55, 56, 57, 58, 59], Sawano et al. [69, 70, 71], Liu et al. [46], Ho [36], and, recently, Alvarado et al. [2, 3, 4]. The Sobolev type inequalities have wide applications in harmonic analysis and partial differential equations (see, for example, [27, 34, 54, 63]).

The ball Banach function space X was introduced by Sawano et al. [68] in order to unify the study of several important function spaces. Compared with Banach function spaces, ball Banach function spaces contain a long list of function spaces. For example, Morrey spaces, Orlicz-slice spaces, mixed-norm Lebesgue spaces, and weighted Lebesgue spaces are all ball Banach function spaces, but they may not be Banach function spaces (see [68, 79, 80] for the details). Recently, Dai et al. [16] studied the Bourgain–Brezis–Mironescu formula of Sobolev type spaces based on ball Banach function spaces. Moreover, the Brezis–Van Schaftingen–Yung formula of Sobolev type spaces based on ball Banach function spaces was also established in [17, 18] and applied to improve fractional Sobolev and Gagliardo–Nirenberg inequalities.

In this article, we establish the fractional Sobolev inequality in the setting of ball Banach function spaces X and, as an application, we characterize the closure of C_{c}^{∞} with respect to $\|\cdot\|_{\dot{W}^{s,X}}$, which is a new Gagliardo semi-norm associated with X. To be precise, assuming that X has the homogeneity property and the non-collapse property, that is, for any $f \in X$, $\lambda \in (0,\infty)$, and $x \in \mathbb{R}^n$, $\|f(\lambda\cdot)\|_X = \lambda^{\alpha}\|f\|_X$ for some $\alpha \in (-\infty,0)$ and $\|\mathbf{1}_{B(x,1)}\|_X \gtrsim 1$ with the implicit positive constant independent of $x \in \mathbb{R}^n$, we show that there exists a positive constant C such that, for any $f \in \dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$,

$$\|f\|_{\dot{W}^{s,X}} := \left\|\left\|\frac{|f(x)-f(y)|}{|x-y|^{s-\alpha}}\right\|_{X(x)}\right\|_{X(y)} \geq C\,\|f\|_{X^{\frac{\alpha}{\alpha+s}}},$$

where $s \in (0,\min\{-\alpha,1\})$. Here and thereafter, for any $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$B(x,r) := \{y \in \mathbb{R}^n : |y-x| < r\}$$

denotes the ball with center x and radius r and, for any $f \in \mathscr{M}$, let

$$\|f(x)\|_{X(x)} := \|f(\cdot)\|_X.$$

Then, using this inequality, we prove that the closure of C_{c}^{∞} with respect to $\|\cdot\|_{\dot{W}^{s,X}}$ modulo constants, denoted by $\mathcal{D}^{s,X}$, is identified with $\dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$. These results have a wide range of applications and, in particular, when X is a Lebesgue space, they reduce to the well-known embeddings, that is, (1.2) and [11, Theorem 3.1]; this indicates that in general the restriction $s \in (0,\min\{-\alpha,1\})$ is sharp. To the best of our knowledge, when X is a Morrey space, a mixed-norm Lebesgue space, the Lebesgue space with power weight, a Besov–Triebel–Lizorkin–Bourgain–Morrey space, or a Lorentz space, these embeddings are new in the literature.

Recall that all the known proofs of (1.2) strongly depend on the explicit integral expression of the Lebesgue norm under consideration. Since $\|\cdot\|_X$ has no explicit expression, the known classical proofs are inapplicable for the ball Banach fractional Sobolev inequality. To overcome this essential difficulty, we fully employ the homogeneity property and the non-collapse property of X, which are used, to replace the dilation invariance and the translation invariance of the Lebesgue norm, respectively; these are crucial tools in the known proofs of the classical fractional Sobolev inequality.

The remainder of this article is organized as follows.

In Section 2, we recall some concepts related to ball Banach function spaces. Then, assuming that a ball Banach function space X has the homogeneity property (Assumption I), we introduce the homogeneous ball Banach fractional Sobolev space $\dot{W}^{s,X}$, thereby extending the concept of the homogeneous fractional Sobolev space $\dot{W}^{s,p}$ to this setting.

Section 3 is devoted to the ball Banach fractional Sobolev inequality. Specifically, in Theorem 3.2, under Assumption I (the homogeneity property) and Assumption II (the non-collapse property of X), we show that, if $s \in (0, \min\{-\alpha, 1\})$, then for any f in $\mathscr{M}_X$ we have $\|f\|_{\dot{W}^{s,X}} \gtrsim \|f\|_{X^{\frac{\alpha}{\alpha+s}}}$ with the implicit positive constant independent of f. This extends the classical fractional Sobolev inequality from the Lebesgue space to the ball Banach function space (see Remark 3.3). Moreover, we prove that the ball Banach fractional Sobolev inequality is valid not only for C_{c}^{∞} functions but also for $\dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$ functions.

Section 4 is devoted to providing an equivalent characterization of the closure of C_{c}^{∞} with respect to $\|\cdot\|_{\dot{W}^{s,X}}$ modulo constants, which is denoted by $\mathcal{D}^{s,X}$. To go further, we need an additional mild assumption on X (see Assumption III). Under Assumptions I, II, and III, we show that $\mathcal{D}^{s,X}$ is identified with $\dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$. To be precise, we prove that there exists a linear isometric isomorphism

$$\mathcal{I}:\ \mathcal{D}^{s,X} \to \dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}.$$

On one hand, using the ball Banach fractional Sobolev inequality, we show that $\mathcal{I}$ is injective. On the other hand, by Hölder's inequality associated with the ball Banach function space (see Lemma 4.8), we prove that $\mathcal{I}$ is surjective. This result is an extension of [11, Theorem 3.1] from the classical Gagliardo semi-norm $\|\cdot\|_{\dot{W}^{s,p}}$ to $\|\cdot\|_{\dot{W}^{s,X}}$ (see Remark 4.2). Finally, we show that Assumption III(iii) is just slightly stronger than a necessary and sufficient condition of $C_{\mathrm{c}}^{\infty} \subset \dot{W}^{s,X}$, which implies that this assumption is necessary in some sense.

In Section 5, we apply our main results obtained in the above sections to several specific examples of ball Banach function spaces, namely the Morrey space M_r^p, the mixed-norm Lebesgue space $L^{\vec{p}}$, the Lebesgue space with power weight L_ω^r, the Besov–Bourgain–Morrey space $\dot{M}B_{q,r}^{p,\tau}$, and the Lorentz space $L^{p,q}$ (see, respectively, Theorems 5.3, 5.6, 5.7, 5.10, 5.11, 5.13, 5.14, 5.16, and 5.17).

Finally, we state our notation and conventions. We let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, and $\mathbb{Z}_+^n := (\mathbb{Z}_+)^n$. We always denote by C a *positive constant* which is independent of the main parameters involved, but it may vary from line to line. The symbol $f \lesssim g$ means that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$. If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g = h$ or $f \lesssim g \le h$. We use $\mathbf{0}$ to denote the *origin* of $\mathbb{R}^n$. For any measurable subset E of $\mathbb{R}^n$, we denote by $\mathbf{1}_E$ its characteristic function and denote by $E^\complement$ its complementary set. In addition, we use the symbol L_{loc}^p with $p \in (0, \infty]$ to denote the set of all measurable functions f on $\mathbb{R}^n$ such that $f\mathbf{1}_E \in L^p$ for any bounded measurable set $E \subset \mathbb{R}^n$. Furthermore, for any $\lambda \in (0, \infty)$ and any ball $B(x, r) \subset \mathbb{R}^n$ with $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $\lambda B(x, r) := B(x, \lambda r)$. Finally, for any $q \in [1, \infty]$, we denote by q' its *conjugate exponent*, that is, $\frac{1}{q} + \frac{1}{q'} = 1$.

2. Preliminaries

In this section, we recall the definition of ball Banach function spaces and introduce homogeneous ball Banach fractional Sobolev spaces. In what follows, we denote by $\mathscr{M}$ the set of all measurable functions on $\mathbb{R}^n$ and we let

$$\mathbb{B} := \{B(x, r) : \ x \in \mathbb{R}^n \text{ and } r \in (0, \infty)\}. \tag{2.1}$$

The following concept is precisely [68, Definition 2.2].

Definition 2.1. Let $X \subset \mathscr{M}$ be a quasi-normed linear space equipped with a quasi-norm $\|\cdot\|_X$, which makes sense for all measurable functions on $\mathbb{R}^n$. Then X is called a *ball quasi-Banach function space* on $\mathbb{R}^n$ if it satisfies:

(i) if $f \in \mathscr{M}$, then $\|f\|_X = 0$ implies that $f = 0$ almost everywhere;
(ii) if $f, g \in \mathscr{M}$, then $|g| \le |f|$ almost everywhere implies that $\|g\|_X \le \|f\|_X$;
(iii) if $\{f_m\}_{m\in\mathbb{N}} \subset \mathscr{M}$ and $f \in \mathscr{M}$, then $0 \le f_m \uparrow f$ almost everywhere as $m \to \infty$ implies that $\|f_m\|_X \uparrow \|f\|_X$ as $m \to \infty$;
(iv) $B \in \mathbb{B}$ implies that $\mathbf{1}_B \in X$, where $\mathbb{B}$ is the same as in (2.1).

Moreover, a ball quasi-Banach function space X is called a *ball Banach function space* if it satisfies:

(v) for any $f, g \in X$, $\|f + g\|_X \le \|f\|_X + \|g\|_X$;
(vi) for any ball $B \in \mathbb{B}$, there exists a positive constant $C_{(B)}$, depending on B, such that, for any $f \in X$, $\int_B |f(x)|\,dx \le C_{(B)}\|f\|_X$.

Remark 2.2. (i) Let X be a ball Banach function space on $\mathbb{R}^n$. By [77, Remark 2.6(i)], we conclude that, for any $f \in \mathscr{M}$, $\|f\|_X = 0$ if and only if $f = 0$ almost everywhere.

(ii) As mentioned in [77, Remark 2.6(ii)], we obtain an equivalent formulation of Definition 2.1 via replacing any ball B by any bounded measurable set E therein.

(iii) We should point out that, in Definition 2.1, if we replace a ball B by any measurable set E with finite measure, we obtain the definition of (quasi-)Banach function spaces, which were originally introduced in [7, Definitions 1.1 and 1.3]. Thus, a (quasi-)Banach function space is also a ball (quasi-)Banach function space and the converse is not necessarily true.

(iv) By [19, Theorem 2], we conclude that both (ii) and (iii) of Definition 2.1 imply that any ball Banach function space is complete.

(v) Examples of Ball Banach function spaces include various function spaces, such as the Lebesgue space L^p, the Morrey space M_r^p, the mixed-norm Lebesgue space $L^{\vec{p}}$, the weighted Lebesgue space L^r_ω, the Besov–Bourgain–Morrey space $\dot{M}B^{p,\tau}_{q,r}$, and the Lorentz space $L^{p,q}$. See, respectively, Definitions 5.2, 5.5, 5.9, 5.12, and 5.15 for their precise definitions and also Section 5 for more details.

The associate space X' of a given ball Banach function space X is defined as follows (see [7, Chapter 1, Section 2] or [68, p. 9]).

Definition 2.3. For any given ball Banach function space X, its *associate space* (also called the *Köthe dual space*) X' is defined by setting

$$X' := \{f \in \mathscr{M} : \ \|f\|_{X'} < \infty\},$$

where, for any $f \in X'$,

$$\|f\|_{X'} := \sup\{\|fg\|_{L^1} : \ g \in X, \ \|g\|_X = 1\}$$

and $\|\cdot\|_{X'}$ is called the *associate norm* of $\|\cdot\|_X$.

Remark 2.4. From [68, Proposition 2.3], we deduce that, if X is a ball Banach function space, then its associate space X' is also a ball Banach function space.

We also recall the concept of the convexity of ball Banach function spaces; this is a part of [68, Definition 2.6].

Definition 2.5. Let X be a ball Banach function space and $p \in (0, \infty)$. The *p-convexification* X^p of X is defined by setting

$$X^p := \{f \in \mathscr{M} : \ |f|^p \in X\},$$

equipped with the *quasi-norm* $\|f\|_{X^p} := \|\,|f|^p\|_X^{1/p}$ for any $f \in X^p$.

We recall the definition of ball Banach function spaces with absolutely continuous norm; see [8, Definition 3.1] and [76, Definition 3.2].

Definition 2.6. A ball Banach function space X is said to have an *absolutely continuous norm* if, for any $f \in X$ and any sequence of measurable sets, $\{E_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ with $E_{j+1} \subset E_j$ for any $j \in \mathbb{N}$ and $\bigcap_{j\in\mathbb{N}} E_j = \emptyset$, $\|f\mathbf{1}_{E_j}\|_X \to 0$ as $j \to \infty$.

Next, we extend the concept of the homogeneous fractional Sobolev space to the ball Banach function space. To this end, we need the following assumption.

Assumption I. *Let X be a ball Banach function space and $\alpha \in (-\infty, 0)$. We consider the* homogeneity property *that for any $\lambda \in (0, \infty)$ and $f \in X$ it holds $\|f(\lambda\cdot)\|_X = \lambda^\alpha\|f\|_X$.*

Remark 2.7. If X satisfies Assumption I with $\alpha \in (-\infty, 0)$, then, by Definition 2.1(iii) and the fact that $\|\mathbf{1}_{B(\mathbf{0},1)}\|_X > 0$ which is a simple consequence of Definition 2.1(i), we conclude that

$$\|\mathbf{1}_{\mathbb{R}^n}\|_X = \lim_{r\to\infty} \left\|\mathbf{1}_{B(\mathbf{0},r)}\right\|_X = \lim_{r\to\infty} r^{-\alpha} \left\|\mathbf{1}_{B(\mathbf{0},1)}\right\|_X = \infty.$$

Definition 2.8. Let X satisfy Assumption I with $\alpha \in (-\infty, 0)$ and let $s \in (0, 1)$. The *homogeneous ball Banach fractional Sobolev space* $\dot{W}^{s,X}$ is defined to be the set of all functions $f \in \mathscr{M}$ such that

$$\|f\|_{\dot{W}^{s,X}} := \left\|\left\|\frac{|f(x) - f(y)|}{|x - y|^{s-\alpha}}\right\|_{X(x)}\right\|_{X(y)} < \infty.$$

Using Remark 2.2(v), we find that L^p, M^p_r, $L^{\vec{p}}$, $L^r_{|\cdot|^\beta}$, $\dot{M}B^{p,\tau}_{q,r}$, and $L^{p,q}$ are all ball Banach function spaces and they can be verified to satisfy Assumption I; see Section 5 for the details. Therefore, we obtain corresponding homogeneous ball Banach fractional Sobolev spaces based on these aforementioned spaces. In particular, when $X := L^p$, $\dot{W}^{s,X}$ reduces to $\dot{W}^{s,p}$ in (1.1).

3. The ball Banach fractional Sobolev inequality

In this section, we establish the fractional Sobolev inequality of the ball Banach fractional Sobolev space, which is called the *ball Banach fractional Sobolev inequality*. In order to achieve this, we need the following non-degeneracy assumption.

Assumption II. *Let X be a ball Banach function space. We say that X has the* non-collapse property *if there exists a positive constant C such that, for any $x \in \mathbb{R}^n$, $\|\mathbf{1}_{B(x,1)}\|_X \geq C$.*

Definition 3.1. Let X be a ball Banach function space. The *space* $\mathscr{M}_X$ is defined to be the set of all functions $f \in \mathscr{M}$ such that, for any $\varepsilon \in (0, \infty)$,

$$\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f(x)|>\varepsilon\}}\right\|_X < \infty.$$

Having established these basic facts, we focus on the main result of this work, which is the following embedding theorem.

Theorem 3.2. *Let X and α satisfy Assumptions I and II and let*

$$s \in (0, \min\{-\alpha, 1\}).$$

Then there exists a positive constant C such that, for any $f \in \mathscr{M}_X$,

$$\|f\|_{X^{\frac{\alpha}{\alpha+s}}} \le C\,\|f\|_{\dot{W}^{s,X}}\,.$$

Remark 3.3. Let $X := L^p$ with $p \in [1, \infty)$ and let $\alpha := -\frac{n}{p}$. In this case, Assumptions I and II obviously hold and hence so does Theorem 3.2, which coincides with the well-known classical fractional Sobolev inequality (1.2). For this reason the range of $s \in (0, \min\{-\alpha, 1\})$ in Theorem 3.2 is sharp in general.

To prove Theorem 3.2, we need the following technical lemma.

Lemma 3.4. *Let X, α, and s be the same as in Theorem 3.2. Then there exists a positive constant C such that for any measurable set $E \subset \mathbb{R}^n$ satisfying $\|\mathbf{1}_E\|_X < \infty$ and for any $x \in E$ we have*

$$\left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{E^\complement}(\cdot)\right\|_X \ge C\|\mathbf{1}_E\|_X^{\frac{s}{\alpha}}. \tag{3.1}$$

Proof. We first consider the case that $E := B(\mathbf{0}, r)$ with $r \in (0, \infty)$. From Assumption I, we deduce that

$$\begin{aligned}\left\|\frac{1}{|\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},r)^\complement}(\cdot)\right\|_X \|\mathbf{1}_{B(\mathbf{0},r)}\|_X^{-\frac{s}{\alpha}} &= r^{s-\alpha}\left\|\frac{1}{|r\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},r)^\complement}(r\cdot)\right\|_X \left\|\mathbf{1}_{B(\mathbf{0},r)}(r\cdot)\right\|_X^{-\frac{s}{\alpha}}\\ &= \left\|\frac{1}{|\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},1)^\complement}(\cdot)\right\|_x \|\mathbf{1}_{B(\mathbf{0},1)}\|_X^{-\frac{s}{\alpha}}.\end{aligned}$$

By Assumption I and the fact that $s \in (0, \min\{-\alpha, 1\})$ we have

$$\begin{aligned}\left\|\frac{1}{|\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},1)^\complement}(\cdot)\right\|_X &\le \sum_{k=1}^{\infty}\left\|\frac{1}{|\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},2^k)\setminus B(\mathbf{0},2^{k-1})}(\cdot)\right\|_X\\ &\le \sum_{k=1}^{\infty} 2^{(k-1)(\alpha-s)}\|\mathbf{1}_{B(\mathbf{0},2^k)}\|_X \sim \sum_{k=1}^{\infty} 2^{-ks} < \infty.\end{aligned}$$

This implies that

$$\left\|\frac{1}{|\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},r)^\complement}(\cdot)\right\|_X \|\mathbf{1}_{B(\mathbf{0},r)}\|_X^{-\frac{s}{\alpha}} = C \in (0, \infty)$$

and hence, for any $r\in(0,\infty)$,

$$\left\|\frac{1}{|\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},r)^\complement}(\cdot)\right\|_X = C\,\|\mathbf{1}_{B(\mathbf{0},r)}\|_X^{\frac{s}{\alpha}}. \tag{3.2}$$

Now, we claim that, for any $r\in(0,\infty)$ and $x\in\mathbb{R}^n$,

$$\left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(x,4r)^\complement}(\cdot)\right\|_X \gtrsim \|\mathbf{1}_{B(x,r)}\|_X^{\frac{s}{\alpha}}. \tag{3.3}$$

We discuss the following two cases based on the size of $|x|$.

Case (i): $|x|\ge 2r$. In this case, let $B(x_1,r_1)$ be a ball with $r_1:=\frac{|x|+4r}{|x|-r}r$ and

$$x_1:=(|x|+4r+r_1)\frac{x}{|x|}=\frac{|x|+4r}{|x|-r}x.$$

It is easy to show that $B(x_1,r_1)\subset B(x,4r)^\complement$. Combining this, Assumption I, and $|x|\ge 2r$, we obtain

$$\begin{aligned}\left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(x,4r)^\complement}(\cdot)\right\|_X \left\|\mathbf{1}_{B(x,r)}\right\|_X^{-\frac{s}{\alpha}} &\ge \left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(x_1,r_1)}(\cdot)\right\|_X \|\mathbf{1}_{B(x,r)}\|_X^{-\frac{s}{\alpha}}\\ &\ge \frac{1}{(4r+2r_1)^{s-\alpha}}\left(\frac{|x|+4r}{|x|-r}\right)^{-\alpha}\|\mathbf{1}_{B(x,r)}\|_X^{\frac{\alpha-s}{\alpha}}\\ &\gtrsim \|\mathbf{1}_{B(\frac{x}{r},1)}\|_X^{\frac{\alpha-s}{\alpha}}.\end{aligned}$$

Using this and Assumption II we conclude that

$$\left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(x,4r)^\complement}(\cdot)\right\|_X \gtrsim \|\mathbf{1}_{B(x,r)}\|_X^{\frac{s}{\alpha}}. \tag{3.4}$$

Case (ii): $|x|<2r$. In this case, from Assumptions I and II, we infer that, for any $r\in(0,\infty)$,

$$r^\alpha\|\mathbf{1}_{B(x,r)}\|_X=\|\mathbf{1}_{B(\frac{x}{r},1)}\|_X\gtrsim 1\sim\|\mathbf{1}_{B(\mathbf{0},6)}\|_X=r^\alpha\|\mathbf{1}_{B(\mathbf{0},6r)}\|_X. \tag{3.5}$$

We observe that $B(x,4r)\subset B(\mathbf{0},6r)$ and for any $y\in B(\mathbf{0},6r)^\complement$ one has $\frac{4}{3}|y|\ge|x-y|$. By this, (3.2), and (3.5), we find that

$$\left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(x,4r)^\complement}(\cdot)\right\|_X \gtrsim \left\|\frac{1}{|\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},6r)^\complement}(\cdot)\right\|_X \sim \|\mathbf{1}_{B(\mathbf{0},6r)}\|_X^{\frac{s}{\alpha}}\gtrsim\|\mathbf{1}_{B(x,r)}\|_X^{\frac{s}{\alpha}}. \tag{3.6}$$

Combining (3.4) and (3.6), we conclude that the above claim holds.

Next, we show that (3.1) is valid. Let

$$r_s:=\sup\left\{r\in[0,\infty):\ \|\mathbf{1}_{B(x,r)\setminus E}\|_X<\|\mathbf{1}_E\|_X\right\}.$$

If $r_s=0$, then, for any $r\in(0,\infty)$, $\|\mathbf{1}_{B(x,r)\setminus E}\|_X\ge\|\mathbf{1}_E\|_X$. From this, we deduce that

$$\begin{aligned}\left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{E^\complement}(\cdot)\right\|_X &\ge \left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(x,r)\setminus E}(\cdot)\right\|_X \ge r^{\alpha-s}\|\mathbf{1}_{B(x,r)\setminus E}\|_X\\ &\ge r^{\alpha-s}\|\mathbf{1}_E\|_X\to\infty\end{aligned}$$

as $r \in (0,\infty)$ and $r \to 0$. This implies that (3.1) holds in this case. If $r_s > 0$, we first prove that $r_s < \infty$. Indeed, when $r \geq (\frac{2\|\mathbf{1}_E\|_X}{\|\mathbf{1}_{B(\mathbf{0},1)}\|_X})^{-\frac{1}{\alpha}} + |x|$, using Minkowski's inequality and Assumption I, we find that

$$\begin{aligned}\|\mathbf{1}_{B(x,r)\setminus E}\|_X &\geq \|\mathbf{1}_{B(x,r)}\|_X - \|\mathbf{1}_E\|_X \geq \|\mathbf{1}_{B(\mathbf{0},r-|x|)}\|_X - \|\mathbf{1}_E\|_X\\ &= (r-|x|)^{-\alpha}\|\mathbf{1}_{B(\mathbf{0},1)}\|_X - \|\mathbf{1}_E\|_X \geq \|\mathbf{1}_E\|_X.\end{aligned}$$

This implies that

$$r_s \leq \left(\frac{2\|\mathbf{1}_E\|_X}{\|\mathbf{1}_{B(\mathbf{0},1)}\|_X}\right)^{-\frac{1}{\alpha}} + |x| < \infty.$$

From the definition of r_s, we further infer that $\|\mathbf{1}_{B(x,\frac{1}{2}r_s)\setminus E}\|_X < \|\mathbf{1}_E\|_X$ and

$$\|\mathbf{1}_{B(x,2r_s)\setminus E}\|_X \geq \|\mathbf{1}_E\|_X.$$

Using these and (3.3), we conclude that

$$\begin{aligned}\left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{E^\complement}(\cdot)\right\|_X &\sim \left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(x,2r_s)\setminus E}(\cdot)\right\|_X + \left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{(B(x,2r_s)\bigcup E)^\complement}(\cdot)\right\|_X\\ &\geq \frac{1}{|2r_s|^{s-\alpha}}\|\mathbf{1}_E\|_X + \left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{(B(x,2r_s)\bigcup E)^\complement}(\cdot)\right\|_X\\ &\geq \left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{E\setminus B(x,2r_s)}(\cdot)\right\|_X + \left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{(B(x,2r_s)\bigcup E)^\complement}(\cdot)\right\|_X\\ &\sim \left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},2r_s)^\complement}(\cdot)\right\|_X \gtrsim \left\|\mathbf{1}_{B(\mathbf{0},\frac{1}{2}r_s)}\right\|_X^{\frac{s}{\alpha}}\\ &\sim \left[\left\|\mathbf{1}_{B(\mathbf{0},\frac{1}{2}r_s)\setminus E}\right\|_X + \left\|\mathbf{1}_{B(\mathbf{0},\frac{1}{2}r_s)\cap E}\right\|_X\right]^{\frac{s}{\alpha}} \gtrsim \|\mathbf{1}_E\|_X^{\frac{s}{\alpha}}.\end{aligned}$$

This finishes the proof of Lemma 3.4. □

Now, we prove Theorem 3.2.

Proof of Theorem 3.2. Notice that $||f(x)|-|f(y)|| \leq |f(x)-f(y)|$ for any $x,y \in \mathbb{R}^n$. Replacing f with $|f|$, without loss of generality, we may only consider the case that $f \geq 0$. Fix $f \geq 0$ and define $D_k := \{x \in \mathbb{R}^n : \ 2^k < f(x) \leq 2^{k+1}\}$ for any $k \in \mathbb{Z}$. It is easy to prove that

$$\|f\|_X \sim \left\|\sum_{i\in\mathbb{Z}} 2^i \mathbf{1}_{D_i}\right\|_X \text{ and } \|f\|_{X^{\frac{\alpha}{\alpha+s}}} \sim \left\|\sum_{i\in\mathbb{Z}} 2^{i\frac{\alpha}{\alpha+s}}\mathbf{1}_{D_i}\right\|_X^{1+\frac{s}{\alpha}}. \tag{3.7}$$

Using Lemma 3.4, we conclude that

$$\begin{aligned}\left\|\left\|\frac{|f(x)-f(y)|}{|x-y|^{s-\alpha}}\right\|_{X(x)}\right\|_{X(y)} &= \left\|\sum_{\substack{i\in\mathbb{Z}\\ |D_i|>0}} \left\|\frac{|f(x)-f(y)|}{|x-y|^{s-\alpha}}\right\|_{X(x)} \mathbf{1}_{D_i}(y)\right\|_{X(y)}\\ &\gtrsim \left\|\sum_{\substack{i\in\mathbb{Z}\\ |D_i|>0}} \left\|\frac{2^i\mathbf{1}_{(D_{i-1}\cup D_i\cup D_{i+1})^c}(x)}{|x-y|^{s-\alpha}}\right\|_{X(x)} \mathbf{1}_{D_i}(y)\right\|_{X(y)}\end{aligned}$$

$$\gtrsim \left\| \sum_{\substack{i\in\mathbb{Z}\\ |D_i|>0}} 2^i \left\| \mathbf{1}_{D_{i-1}\cup D_i\cup D_{i+1}}(x) \right\|_{X(x)}^{\frac{s}{\alpha}} \mathbf{1}_{D_i}(y) \right\|_{X(y)} .$$

Hence,

$$\begin{aligned}
&\left\| \left\| \frac{|f(x)-f(y)|}{|x-y|^{s-\alpha}} \right\|_{X(x)} \right\|_{X(y)} \left\| \sum_{k\in\mathbb{Z}} 2^{k\frac{\alpha}{\alpha+s}} \mathbf{1}_{D_k} \right\|_X^{-\frac{s}{\alpha}} \\
&\quad \gtrsim \left\| \sum_{\substack{i\in\mathbb{Z}\\ |D_i|>0}} 2^i \left\| \sum_{k\in\mathbb{Z}} 2^{k\frac{\alpha}{\alpha+s}} \mathbf{1}_{D_k} \right\|_X^{-\frac{s}{\alpha}} \left\| \mathbf{1}_{D_{i-1}\cup D_i\cup D_{i+1}}(x) \right\|_{X(x)}^{\frac{s}{\alpha}} \mathbf{1}_{D_i}(y) \right\|_{X(y)} \\
&\quad \gtrsim \left\| \sum_{i\in\mathbb{Z}} 2^{i\frac{\alpha}{\alpha+s}} \mathbf{1}_{D_i} \right\|_X ,
\end{aligned}$$

which, together with (3.7), further implies that

$$\left\| \left\| \frac{|f(x)-f(y)|}{|x-y|^{s-\alpha}} \right\|_{X(x)} \right\|_{X(y)} \gtrsim \left\| \sum_{i\in\mathbb{Z}} 2^{i\frac{\alpha}{\alpha+s}} \mathbf{1}_{D_i} \right\|_X^{1+\frac{s}{\alpha}} \sim \|f\|_{X^{\frac{\alpha}{\alpha+s}}} .$$

This finishes the proof of Theorem 3.2. □

As a corollary of Theorem 3.2, we have the following conclusion.

Corollary 3.5. *Let X and α satisfy Assumptions I and II and let*

$$s \in (0, \min\{-\alpha, 1\}).$$

Then there exists a positive constant C such that, for any $f \in C_c^\infty$,

$$\|f\|_{X^{\frac{\alpha}{\alpha+s}}} \le C \|f\|_{\dot{W}^{s,X}} .$$

Proof. Let $f \in C_c^\infty$. Then, for any $\varepsilon \in (0,\infty)$, $\{x \in \mathbb{R}^n : |f(x)| > \varepsilon\} \subset \operatorname{supp}(f)$ and there exists $r \in (0,\infty)$ such that $\operatorname{supp}(f) \subset B(\mathbf{0}, r)$. From Definition 2.1(ii) and (iv), we infer that

$$\left\| \mathbf{1}_{\{x\in\mathbb{R}^n:\ |f(x)|>\varepsilon\}} \right\|_X \le \left\| \mathbf{1}_{B(\mathbf{0},r)} \right\|_X < \infty$$

and hence $f \in \mathscr{M}_X$, which implies $C_c^\infty \subset \mathscr{M}_X$. Combining this and Theorem 3.2, we complete the proof of Corollary 3.5. □

By this, we have proved that the ball Banach fractional Sobolev inequality holds for any $f \in C_c^\infty$. To extend the ball Banach fractional Sobolev inequality to a wider class $\dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$ requires a considerable amount of additional work; for this purpose we need the following two technical lemmas.

Lemma 3.6. *Let X and α satisfy Assumptions I and II, $s \in (0, \min\{-\alpha, 1\})$, $m \in (0,\infty)$, and $-\infty < b < a < \infty$. Then there exists a positive constant C such that for any real-valued function $f \in \mathscr{M}$ that satisfies*

$$\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)>a\}}\right\|_X > m \ \text{ and } \ \left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)<b\}}\right\|_X > m \tag{3.8}$$

we have

$$\|f\|_{\dot{W}^{s,X}} \geq C m^{\frac{\alpha+s}{\alpha}}(a-b). \tag{3.9}$$

Proof. Let us fix a real-valued function $f \in \mathscr{M}$ that satisfies (3.8). From Definition 2.1(iii) and (3.8), we deduce that there exists $r \in (0,\infty)$ such that

$$\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)>a\}\cap B(\mathbf{0},r)}\right\|_X > m \ \text{ and } \ \left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)<b\}\cap B(\mathbf{0},r)}\right\|_X > m. \tag{3.10}$$

Let $A_1 := \{x \in B(\mathbf{0}, r):\ f(x) > 2a/3 + b/3\}$,

$$A_2 := \{x \in B(\mathbf{0}, r):\ a/3 + 2b/3 < f(x) \leq 2a/3 + b/3\},$$

and $A_3 := \{x \in B(\mathbf{0}, r):\ f(x) \leq a/3 + 2b/3\}$. For any measurable set $E \subset B(\mathbf{0}, r)$ and $x \in E$, let

$$r_x^{(E)} := \sup\left\{r_s \in [0,\infty):\ \left\|\mathbf{1}_{[B(\mathbf{0},r)\cap B(x,r_s)]\setminus E}\right\|_X < \|\mathbf{1}_E\|_X\right\}. \tag{3.11}$$

For any $i \in \{1, 2, 3\}$, we define $r_i := \sup\{r_x^{(A_i)}:\ x \in A_i\}$.

If $\min\{r_1, r_3\} > \frac{1}{8}r$, then there exist $x_1 \in A_1$ satisfying $r_{x_1}^{(A_1)} > \frac{1}{8}r$ and $x_3 \in A_3$ satisfying $r_{x_3}^{(A_3)} > \frac{1}{8}r$. It is easy to show that there exists x_i^* such that $B(x_i^*, \frac{1}{16}r) \subset B(x_i, \frac{1}{8}r) \cap B(\mathbf{0}, r)$ for any $i \in \{1, 3\}$. By this, (3.11), the definition of A_3, (3.10), Assumptions I and II, we conclude that

$$\begin{aligned}
&\left\|\left\|\frac{|f(x)-f(y)|}{|x-y|^{s-\alpha}}\right\|_{X(x)}\right\|_{X(y)} \\
&\quad\geq \left\|\left\|\frac{|f(x)-f(y)|}{|x-y|^{s-\alpha}}\mathbf{1}_{A_1}(x)\right\|_{X(x)}\mathbf{1}_{A_3}(y)\right\|_{X(y)} \\
&\quad\gtrsim \frac{a-b}{r^{s-\alpha}}\left\|\mathbf{1}_{B(x_1,\frac{1}{8}r)\cap B(\mathbf{0},r)}\right\|_X \left\|\mathbf{1}_{B(x_3,\frac{1}{8}r)\cap B(\mathbf{0},r)}\right\|_X^{-\frac{s}{\alpha}} \left\|\mathbf{1}_{A_3}\right\|_X^{\frac{\alpha+s}{\alpha}} \\
&\quad\geq \frac{a-b}{r^{s-\alpha}}\left\|\mathbf{1}_{B(x_1^*,\frac{1}{16}r)}\right\|_X \left\|\mathbf{1}_{B(x_3^*,\frac{1}{16}r)}\right\|_X^{-\frac{s}{\alpha}} \left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)<b\}\cap B(\mathbf{0},r)}\right\|_X^{\frac{\alpha+s}{\alpha}} \\
&\quad\gtrsim m^{\frac{\alpha+s}{\alpha}}(a-b)\left\|\mathbf{1}_{B(\frac{16}{r}x_1^*,1)}\right\|_X \left\|\mathbf{1}_{B(\frac{16}{r}x_3^*,1)}\right\|_X^{-\frac{s}{\alpha}} \gtrsim m^{\frac{\alpha+s}{\alpha}}(a-b).
\end{aligned} \tag{3.12}$$

If $\min\{r_1, r_3\} \leq \frac{1}{8}r$, without loss of generality, we may assume $r_1 \leq \frac{1}{8}r$. We first claim that, for any $x \in B(\mathbf{0}, r)$ and $\tilde{r} \leq \frac{1}{8}r$,

$$\left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},r)\setminus B(x,2\tilde{r})}(\cdot)\right\|_X \gtrsim \left\|\mathbf{1}_{B(\mathbf{0},r)\cap B(x,\frac{1}{2}\tilde{r})}\right\|_X^{\frac{s}{\alpha}}. \tag{3.13}$$

To prove this, we consider the following two cases for the size of $|x|$.

Case (i): $|x| \geq 4\tilde{r}$. In this case, let $r_1 := \frac{|x|-2\tilde{r}}{|x|+\tilde{r}}\tilde{r}$ and

$$x_1 := (|x| - 2\tilde{r} - r_1)\frac{x}{|x|} = \frac{|x|-2\tilde{r}}{|x|+\tilde{r}}x.$$

It is easy to show $B(x_1, r_1) \subset B(\mathbf{0}, r) \setminus B(x, 2\tilde{r})$. Combining this and Assumption I, we obtain

$$\begin{aligned}
&\left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},r)\setminus B(x,2\tilde{r})}(\cdot)\right\|_X \left\|\mathbf{1}_{B(\mathbf{0},r)\cap B(x,\frac{1}{2}\tilde{r})}\right\|_X^{-\frac{s}{\alpha}}\\
&\quad\geq \left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(x_1,r_1)}(\cdot)\right\|_X \left\|\mathbf{1}_{B(\mathbf{0},r)\cap B(x,\frac{1}{2}\tilde{r})}\right\|_X^{-\frac{s}{\alpha}}\\
&\quad\geq \frac{1}{(\tilde{r}+2r_1)^{s-\alpha}}\left(\frac{|x|-2\tilde{r}}{|x|+\tilde{r}}\right)^{-\alpha}\left\|\mathbf{1}_{B(x,\tilde{r})}\right\|_X \left\|\mathbf{1}_{B(\mathbf{0},r)\cap B(x,\frac{1}{2}\tilde{r})}\right\|_X^{-\frac{s}{\alpha}}\\
&\quad\gtrsim \frac{1}{\tilde{r}^{s-\alpha}}\left\|\mathbf{1}_{B(x,\tilde{r})}\right\|_X \left\|\mathbf{1}_{B(\frac{|x|-\frac{1}{4}\tilde{r}}{|x|}x,\frac{1}{4}\tilde{r})}\right\|_X^{-\frac{s}{\alpha}} \gtrsim \left\|\mathbf{1}_{B(\frac{x}{\tilde{r}},1)}\right\|_X \left\|\mathbf{1}_{B(\frac{4|x|-\tilde{r}}{|x|\tilde{r}}x,1)}\right\|_X^{-\frac{s}{\alpha}}.
\end{aligned}$$

By this and Assumption II, we conclude that (3.13) holds in this case.

Case (ii): $|x| < 4\tilde{r}$. In this case, from $\widetilde{r} \leq \frac{1}{8}r$ and Assumption I, we deduce that

$$\begin{aligned}
&\left\|\frac{1}{|\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},r)\setminus B(\mathbf{0},6\tilde{r})}(\cdot)\right\|_X \|\mathbf{1}_{B(\mathbf{0},6\tilde{r})}\|_X^{-\frac{s}{\alpha}}\\
&\quad\geq \left\|\frac{1}{|\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},8\tilde{r})\setminus B(\mathbf{0},6\tilde{r})}(\cdot)\right\|_X \|\mathbf{1}_{B(\mathbf{0},6\tilde{r})}\|_X^{-\frac{s}{\alpha}}\\
&\quad= \left\|\frac{1}{|\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},\frac{4}{3})\setminus B(\mathbf{0},1)}(\cdot)\right\|_X \|\mathbf{1}_{B(\mathbf{0},1)}\|_X^{-\frac{s}{\alpha}} > 0.
\end{aligned}\tag{3.14}$$

It is easy to prove that $B(x, 2\tilde{r}) \subset B(\mathbf{0}, 6\tilde{r})$ and, for any $y \in B(\mathbf{0}, 6\tilde{r})^{\complement}$, $\frac{5}{3}|y| \geq |x-y|$. By this, (3.14), and an argument similar to that used in the estimation of (3.5), we find that

$$\begin{aligned}
\left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},r)\setminus B(x,2\tilde{r})}(\cdot)\right\|_X &\gtrsim \left\|\frac{1}{|\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},r)\setminus B(\mathbf{0},6\tilde{r})}(\cdot)\right\|_X\\
&\gtrsim \|\mathbf{1}_{B(\mathbf{0},6\tilde{r})}\|_X^{\frac{s}{\alpha}} \gtrsim \|\mathbf{1}_{B(x,\frac{1}{2}\tilde{r})}\|_X^{\frac{s}{\alpha}}.
\end{aligned}$$

This shows (3.13) in this case. Altogether, we conclude that the above claim holds.

By (3.11) and (3.13), we find that, for any measurable set $E \subset B(\mathbf{0}, r)$ and any $x \in E$ satisfying $r_x^{(E)} \in (0, \frac{1}{8}r)$,

$$\begin{aligned}
&\left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},r)\setminus E}(\cdot)\right\|_X\\
&\quad\sim \left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},r)\setminus[E\cup B(x,2r_x^{(E)})]}(\cdot)\right\|_X + \left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{[B(\mathbf{0},r)\cap B(x,2r_x^{(E)})]\setminus E}(\cdot)\right\|_X\\
&\quad\geq \left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},r)\setminus[E\cup B(x,2r_x^{(E)})]}(\cdot)\right\|_X + \left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{E\setminus B(x,2r_x^{(E)})}(\cdot)\right\|_X\\
&\quad\sim \left\|\frac{1}{|x-\cdot|^{s-\alpha}}\mathbf{1}_{B(\mathbf{0},r)\setminus B(x,2r_x^{(E)})}(\cdot)\right\|_X \gtrsim \left\|\mathbf{1}_{B(\mathbf{0},r)\cap B(x,\frac{1}{2}r_x^{(E)})}\right\|_X^{\frac{s}{\alpha}}
\end{aligned}\tag{3.15}$$

$$\sim \left\{\left\|\mathbf{1}_{B(\mathbf{0},r)\cap B(x,\frac{1}{2}r_x^{(E)})\cap E}\right\|_X + \left\|\mathbf{1}_{B(\mathbf{0},r)\cap B(x,\frac{1}{2}r_x^{(E)})\setminus E}\right\|_X\right\}^{\frac{s}{\alpha}} \gtrsim \|\mathbf{1}_E\|_X^{\frac{s}{\alpha}}.$$

Let $g := (f - 2a/3 - b/3)^+$. Here and thereafter, for any $a \in \mathbb{R}$, we let $a^+ := \max\{a, 0\}$. It is easy to prove that, for any $x, y \in \mathbb{R}^n$, $|f(x) - f(y)| \geq |g(x) - g(y)|$ and hence

$$\left\|\left\|\frac{|f(x)-f(y)|}{|x-y|^{s-\alpha}}\right\|_{X(x)}\right\|_{X(y)} \geq \left\|\left\|\frac{|g(x)-g(y)|}{|x-y|^{s-\alpha}}\right\|_{X(x)}\right\|_{X(y)}. \tag{3.16}$$

Let $D_k := \{x \in B(\mathbf{0}, r) :\ 2^k < g(x) \leq 2^{k+1}\}$ for any $k \in \mathbb{Z}$. By standard arguments we have

$$\left\|g\mathbf{1}_{B(\mathbf{0},r)}\right\|_X \sim \left\|\sum_{i\in\mathbb{Z}} 2^i \mathbf{1}_{D_i}\right\|_X \text{ and } \left\|g\mathbf{1}_{B(\mathbf{0},r)}\right\|_{X^{\frac{\alpha}{\alpha+s}}} \sim \left\|\sum_{i\in\mathbb{Z}} 2^{i\frac{\alpha}{\alpha+s}} \mathbf{1}_{D_i}\right\|_X^{1+\frac{s}{\alpha}}. \tag{3.17}$$

From the assumption that $r_1 \leq \frac{1}{8}r$, we deduce that, for any $i \in \mathbb{Z}$ and any $x \in D_{i-1} \cup D_i \cup D_{i+1}$, $r_x^{(D_{i-1}\cup D_i\cup D_{i+1})} \in (0, \frac{1}{8}r]$. Using this and (3.15), we conclude that

$$\begin{aligned}
\left\|\left\|\frac{|g(x)-g(y)|}{|x-y|^{s-\alpha}}\right\|_{X(x)}\right\|_{X(y)} &\geq \left\|\sum_{i\in\mathbb{Z}} \left\|\frac{|g(x)-g(y)|}{|x-y|^{s-\alpha}}\right\|_{X(x)} \mathbf{1}_{D_i}(y)\right\|_{X(y)} \\
&\gtrsim \left\|\sum_{\substack{i\in\mathbb{Z}\\ |D_i|>0}} \left\|\frac{2^i \mathbf{1}_{B(\mathbf{0},r)\setminus(D_{i-1}\cup D_i\cup D_{i+1})}(x)}{|x-y|^{s-\alpha}}\right\|_{X(x)} \mathbf{1}_{D_i}(y)\right\|_{X(y)} \\
&\gtrsim \left\|\sum_{\substack{i\in\mathbb{Z}\\ |D_i|>0}} 2^i \left\|\mathbf{1}_{D_{i-1}\cup D_i\cup D_{i+1}}(x)\right\|_{X(x)}^{\frac{s}{\alpha}} \mathbf{1}_{D_i}(y)\right\|_{X(y)}.
\end{aligned}$$

By this, we immediately have

$$\begin{aligned}
&\left\|\left\|\frac{|g(x)-g(y)|}{|x-y|^{s-\alpha}}\right\|_{X(x)}\right\|_{X(y)} \left\|\sum_{i\in\mathbb{Z}} 2^{i\frac{\alpha}{\alpha+s}} \mathbf{1}_{D_i}\right\|_X^{-\frac{s}{\alpha}} \\
&\quad\gtrsim \left\|\sum_{\substack{i\in\mathbb{Z}\\ |D_i|>0}} 2^i \left\|\sum_{i\in\mathbb{Z}} 2^{i\frac{\alpha}{\alpha+s}} \mathbf{1}_{D_i}(x)\right\|_{X(x)}^{-\frac{s}{\alpha}} \left\|\mathbf{1}_{D_{i-1}\cup D_i\cup D_{i+1}}(x)\right\|_{X(x)}^{\frac{s}{\alpha}} \mathbf{1}_{D_i}(y)\right\|_{X(y)} \\
&\quad\gtrsim \left\|\sum_{i\in\mathbb{Z}} 2^{i\frac{\alpha}{\alpha+s}} \mathbf{1}_{D_i}\right\|_X.
\end{aligned}$$

Combining this, (3.16), (3.17), and (3.10), we find that

$$\begin{aligned}
\left\|\left\|\frac{|f(x)-f(y)|}{|x-y|^{s-\alpha}}\right\|_{X(x)}\right\|_{X(y)} &\geq \left\|\left\|\frac{|g(x)-g(y)|}{|x-y|^{s-\alpha}}\right\|_{X(x)}\right\|_{X(y)} \\
&\gtrsim \left\|\sum_{i\in\mathbb{Z}} 2^{i\frac{\alpha}{\alpha+s}} \mathbf{1}_{D_i}\right\|_X^{1+\frac{s}{\alpha}} \sim \left\|g\mathbf{1}_{B(\mathbf{0},r)}\right\|_{X^{\frac{\alpha}{\alpha+s}}} \\
&\gtrsim (a-b) \left\|\mathbf{1}_{B(\mathbf{0},r)\cap\{x\in\mathbb{R}^n:\ f(x)>a\}}\right\|_X^{\frac{\alpha+s}{\alpha}}
\end{aligned} \tag{3.18}$$

$$\gtrsim m^{\frac{\alpha+s}{\alpha}}(a-b),$$

thus, (3.9) also holds in Case (ii). Now, (3.12) and (3.18) complete the proof of Lemma 3.6. □

Lemma 3.7. *Let $f \in \dot{W}^{s,X}$. Then there exists a constant $C \in \mathbb{C}$ such that $f - C \in \mathscr{M}_X$.*

Proof. Without loss of generality, we may assume that f is a real-valued function, otherwise we can consider the real part and the imaginary part of f separately. Let

$$I := \sup\left\{\lambda \in \mathbb{R} : \left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)\geq\lambda\}}\right\|_X = \infty\right\}$$

and

$$i := \inf\left\{\lambda \in \mathbb{R} : \left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)\leq\lambda\}}\right\|_X = \infty\right\}.$$

We first prove that $I = i$. Assume that $I < i$, and hence there exists $\lambda_1 \in (I, i)$. By Remark 2.7 and the definitions of I and i, we conclude that

$$\infty = \|\mathbf{1}_{\mathbb{R}^n}\|_X \leq \left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)\geq\lambda_1\}}\right\|_X + \left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)\leq\lambda_1\}}\right\|_X < \infty,$$

which is a contradiction. Assume that $I > i$ and then there exist constants λ_2 and λ_3 satisfying $i < \lambda_2 < \lambda_3 < I$. By the definitions of I and i, we have

$$\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)>\lambda_3\}}\right\|_X = \infty = \left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)<\lambda_2\}}\right\|_X.$$

From this and Lemma 3.6, we infer that, for any $m \in (0, \infty)$,

$$\|f\|_{\dot{W}^{s,X}} \geq C m^{\frac{\alpha+s}{\alpha}}(\lambda_3 - \lambda_2), \tag{3.19}$$

where C is the same as in (3.9). By the arbitrariness of m, we conclude that (3.19) contradicts the assumption that $f \in \dot{W}^{s,X}$. This shows $I = i$.

Now, we prove that $I \in \mathbb{R}$. In order to show this, we assume that $I = \infty = i$ or $I = -\infty = i$ and we argue by contradiction. We only consider the first case because the argument for the second case is similar. By Definition 2.1(iii) and Remark 2.7, we conclude that

$$\lim_{\lambda\to\infty} \left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)<\lambda\}}\right\|_X = \|\mathbf{1}_{\mathbb{R}^n}\|_X = \infty.$$

From this, we deduce that there exists a constant $\lambda \in (-\infty, \infty)$ such that, for any $m \in (0, \infty)$,

$$\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)<\lambda\}}\right\|_X > m. \tag{3.20}$$

Then, using the definition of i, we find that $\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)\leq\lambda\}}\|_X < \infty$ and

$$\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)\leq\lambda+1\}}\|_X < \infty$$

and hence

$$\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)>\lambda+1\}}\|_X \geq \|\mathbf{1}_{\mathbb{R}^n}\|_X - \|\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x)\leq\lambda+1\}}\|_X = \infty. \tag{3.21}$$

By (3.20), (3.21), and Lemma 3.6, we conclude that for any $s \in (0, \min\{-\alpha, 1\})$ and $m \in (0, \infty)$ we have $\|f\|_{\dot{W}^{s,X}} \gtrsim m^{\frac{\alpha+s}{\alpha}}$. From the arbitrariness of m, we infer that $\|f\|_{\dot{W}^{s,X}} = \infty$, which contradicts the assumption that $f \in \dot{W}^{s,X}$. Thus, $I = i \in (-\infty, \infty)$. By the definitions of i and I, we obtain, for any $\varepsilon \in (0, \infty)$, $\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f(x)-I|>\varepsilon\}}\|_X < \infty$, which completes the proof of Lemma 3.7. □

Using the above lemmas, we conclude the following corollary.

Corollary 3.8. *Let X and α satisfy Assumptions I and II and let*

$$s \in (0, \min\{-\alpha, 1\}).$$

Then there exists a positive constant $\widetilde{C}$ such that for any $f \in \dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$ we have $\|f\|_{X^{\frac{\alpha}{\alpha+s}}} \leq \widetilde{C}\|f\|_{\dot{W}^{s,X}}$.

Proof. Let $f \in \dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$. From Lemma 3.7, there exists a constant $C \in \mathbb{C}$ such that, for any $\varepsilon \in (0, \infty)$, $\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f(x)-C|>\varepsilon\}}\|_X < \infty$, that is, $f - C \in \mathscr{M}_X$. Assuming $C \neq 0$ and letting $\varepsilon := \frac{|C|}{2}$, we obtain

$$\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f(x)-C|>\frac{|C|}{2}\}}\right\|_X < \infty. \tag{3.22}$$

On the other hand, using $f \in X^{\frac{\alpha}{\alpha+s}}$, we find that

$$\left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f(x)|>\frac{|C|}{2}\}}\right\|_X \leq \left(\frac{2}{|C|}\|f\|_{X^{\frac{\alpha}{\alpha+s}}}\right)^{\frac{\alpha}{\alpha+s}} < \infty. \tag{3.23}$$

By Remark 2.7, (3.22), and (3.23), we conclude that

$$\infty = \|\mathbf{1}_{\mathbb{R}^n}\|_X \leq \left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f(x)-C|>\frac{|C|}{2}\}}\right\|_X + \left\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |f(x)|>\frac{|C|}{2}\}}\right\|_X < \infty,$$

which is a contradiction. Thus, $C = 0$ and $f \in \mathscr{M}_X$, which implies $\dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}} \subset \mathscr{M}_X$. Combining this and Theorem 3.2, we then complete the proof of Corollary 3.8. □

4. Closure of C_c^∞ with respect to $\|\cdot\|_{\dot{W}^{s,X}}$

In this section, we characterize the closure of C_c^∞ with respect to $\|\cdot\|_{\dot{W}^{s,X}}$. Notice that, for any $C \in \mathbb{C}$, $\|f + C\|_{\dot{W}^{s,X}} = \|f\|_{\dot{W}^{s,X}}$. Thus, it makes sense to define the space of equivalence classes

$$\mathcal{D}^{s,X} := \left\{[f] :\ f \in \overline{C_c^\infty}^{\|\cdot\|_{\dot{W}^{s,X}}}\right\}$$

with the norm $\|\,[f]\,\|_{\mathcal{D}^{s,X}} := \|f\|_{\dot{W}^{s,X}}$, where $[f] := \{f + C :\ C \in \mathbb{C}\}$.

Next, we show the space $\mathcal{D}^{s,X}$ is identified with $\dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$. This identification relies on certain natural assumptions that are valid on most important examples of ball Banach function spaces.

Assumption III. *Let X and α satisfy Assumption I and let $s \in (0, \min\{-\alpha, 1\})$. Assume that*

(i) *X has an absolutely continuous norm (see Definition 2.6);*

(ii) *there exists a positive constant C such that, for any $r \in (0, \infty)$ and $f \in \mathscr{M}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying $\| \|f(x,y)\|_{X(x)}\|_{X(y)} < \infty$,*
$$\left\| \left\| \fint_{B(\mathbf{0},r)} f(x-z, y-z)\, dz \right\|_{X(x)} \right\|_{X(y)} \le C \left\| \|f(x,y)\|_{X(x)} \right\|_{X(y)};$$

(iii) *there exists a positive constant C such that, for any $x \in B(\mathbf{0}, 1)$,*
$$\left\| \frac{\mathbf{1}_{B(x,1)}(\cdot)}{|x - \cdot|^{s-\alpha-1}} \right\|_X < C.$$

Theorem 4.1. *Let X and α satisfy Assumption I, $s \in (0, \min\{-\alpha, 1\})$, and X also satisfy Assumptions II and III. Then there exists a linear isometric isomorphism*
$$\mathcal{I} : \mathcal{D}^{s,X} \to \dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}.$$
In other words, the space $\mathcal{D}^{s,X}$ is identified with $\dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$.

Remark 4.2. Let $X := L^p$ with $p \in [1, \infty)$ and let $\alpha := -\frac{n}{p}$. In this case, Assumptions I, II, and III hold and hence so does Theorem 4.1, which coincides with [11, Theorem 3.1].

The proof of Theorem 4.1 is based on the following technical lemmas.

Lemma 4.3. *Let X and α satisfy Assumption I and let $s \in (0, \min\{-\alpha, 1\})$. Then $\dot{W}^{s,X}$ contains C_c^∞ if and only if X satisfies*
$$\left\| \left\| \frac{\mathbf{1}_{B(y,1)}(x)}{|x-y|^{s-\alpha-1}} \right\|_{X(x)} \mathbf{1}_{B(\mathbf{0},1)}(y) \right\|_{X(y)} < \infty. \tag{4.1}$$

Proof. We first show the sufficiency. Assume that (4.1) holds. Let $f \in C_c^\infty$ satisfy $\operatorname{supp}(f) \subset B(\mathbf{0}, r)$ with $r \in (0, \infty)$. From this, Assumption I, and $s \in (0, \min\{-\alpha, 1\})$, we infer that, for any $y \in B(\mathbf{0}, 2r)$,
$$\begin{aligned}
&\left\| \frac{|f(\cdot) - f(y)|}{|\cdot - y|^{s-\alpha}} \right\|_X \\
&\quad\le \left\| \frac{|f(\cdot) - f(y)|}{|\cdot - y|^{s-\alpha}} \mathbf{1}_{B(\mathbf{0},4r)} \right\|_X + \sum_{k=1}^{\infty} \left\| \frac{|f(\cdot) - f(y)|}{|\cdot - y|^{s-\alpha}} \mathbf{1}_{B(\mathbf{0},2^{k+2}r)\setminus B(\mathbf{0},2^{k+1}r)} \right\|_X \\
&\quad\lesssim \|\nabla f\|_{L^\infty} \left\| \frac{\mathbf{1}_{B(\mathbf{0},4r)}(\cdot)}{|\cdot - y|^{s-\alpha-1}} \right\|_X + \|f\|_{L^\infty} \sum_{k=1}^{\infty} \left\| \frac{\mathbf{1}_{B(\mathbf{0},2^{k+2}r)\setminus B(\mathbf{0},2^{k+1}r)}(\cdot)}{|\cdot|^{s-\alpha}} \right\|_X \\
&\quad\lesssim \left\| \frac{\mathbf{1}_{B(y,1)}(\cdot)}{|\cdot - y|^{s-\alpha-1}} \right\|_X + \left\| \frac{\mathbf{1}_{B(\mathbf{0},4r)\setminus B(y,1)}(\cdot)}{|\cdot - y|^{s-\alpha-1}} \right\|_X + \sum_{k=1}^{\infty} 2^{-sk} \left\| \mathbf{1}_{B(\mathbf{0},8r)\setminus B(\mathbf{0},4r)} \right\|_X \\
&\quad\lesssim \left\| \frac{\mathbf{1}_{B(y,1)}(\cdot)}{|\cdot - y|^{s-\alpha-1}} \right\|_X + \|\mathbf{1}_{B(\mathbf{0},4r)}\| + \sum_{k=1}^{\infty} 2^{-sk} \left\| \mathbf{1}_{B(\mathbf{0},8r)\setminus B(\mathbf{0},4r)} \right\|_X
\end{aligned} \tag{4.2}$$

$$\lesssim \left\| \frac{\mathbf{1}_{B(y,1)}(\cdot)}{|\cdot - y|^{s-\alpha-1}} \right\|_X + 1$$

and, by the support condition of f, for any $y \in B(\mathbf{0}, 2r)^{\complement}$,

$$\left\| \frac{|f(\cdot) - f(y)|}{|\cdot - y|^{s-\alpha}} \right\|_X \lesssim \frac{1}{|y|^{s-\alpha}} \|f\|_{L^\infty} \lesssim \frac{1}{|y|^{s-\alpha}}. \tag{4.3}$$

Using Assumption I, we find that

$$\begin{aligned}
&\left\| \left\| \frac{\mathbf{1}_{B(y,1)}(x)}{|x-y|^{s-\alpha-1}} \right\|_{X(x)} \mathbf{1}_{B(\mathbf{0},2r)}(y) \right\|_{X(y)} \\
&\quad = (2r)^{-2\alpha} \left\| \left\| \frac{\mathbf{1}_{B(2ry,1)}(2rx)}{|2rx - 2ry|^{s-\alpha-1}} \right\|_{X(x)} \mathbf{1}_{B(\mathbf{0},2r)}(2ry) \right\|_{X(y)} \\
&\quad = (2r)^{1-s-\alpha} \left\| \left\| \frac{\mathbf{1}_{B(y,(2r)^{-1})}(x)}{|x-y|^{s-\alpha-1}} \right\|_{X(x)} \mathbf{1}_{B(\mathbf{0},1)}(y) \right\|_{X(y)}.
\end{aligned}$$

Combining this, (4.2), (4.3), Assumption I, (4.1), and $s \in (0, \min\{-\alpha, 1\})$, we conclude that

$$\begin{aligned}
\|f\|_{\dot{W}^{s,X}} &\le \left\| \left\| \frac{|f(x) - f(y)|}{|x-y|^{s-\alpha}} \right\|_{X(x)} \mathbf{1}_{B(\mathbf{0},2r)}(y) \right\|_{X(y)} \\
&\quad + \left\| \left\| \frac{|f(x) - f(y)|}{|x-y|^{s-\alpha}} \right\|_{X(x)} \mathbf{1}_{B(\mathbf{0},2r)^{\complement}}(y) \right\|_{X(y)} \\
&\lesssim \left\| \left\| \frac{\mathbf{1}_{B(y,1)}(x)}{|x-y|^{s-\alpha-1}} \right\|_{X(x)} \mathbf{1}_{B(\mathbf{0},2r)}(y) \right\|_{X(y)} + \|\mathbf{1}_{B(\mathbf{0},2r)}\|_{X(y)} \\
&\quad + \sum_{k=1}^{\infty} \left\| \frac{\mathbf{1}_{B(\mathbf{0},2^{k+1}r)\setminus B(\mathbf{0},2^k r)}(y)}{|y|^{s-\alpha}} \right\|_{X(y)} \\
&\lesssim \max\left\{1, (2r)^{1-s-\alpha}\right\} \left\| \left\| \frac{\mathbf{1}_{B(y,1)}(x)}{|x-y|^{s-\alpha-1}} \right\|_{X(x)} \mathbf{1}_{B(\mathbf{0},1)}(y) \right\|_{X(y)} + \left\|\mathbf{1}_{B(\mathbf{0},2r)}\right\|_{X(y)} \\
&\quad + \sum_{k=1}^{\infty} 2^{-sk} \left\|\mathbf{1}_{B(\mathbf{0},4r)\setminus B(\mathbf{0},2r)}(x)\right\|_{X(x)} \\
&< \infty.
\end{aligned}$$

This finishes the proof of the sufficiency.

Now we prove the necessity. Let $\vec{e_i} := (0, \dots, 0, 1, 0, \dots, 0)$ (the ith entry is 1 and the other entries are 0) for any $i \in \{1, \dots, n\}$. We first claim that, for any $x, y \in B(\mathbf{0}, 2)$,

$$\max\left\{\left| |x - 4n\vec{e_i}| - |y - 4n\vec{e_i}| \right| : \ i \in \{1, \dots, n\}\right\} \gtrsim |x - y|.$$

To see this, let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Assume that

$$|x_1 - y_1| = \max\{|x_i - y_i| : \ i \in \{1, \dots, n\}\}$$

and $x_1 < y_1$. By $x, y \in B(\mathbf{0}, 2)$, we find that

$$\begin{aligned}(4.4)\qquad &\left|\left|x - 4n\vec{e_1}\right| - \left|y - 4n\vec{e_1}\right|\right| \\ &\geq \left[(4n - x_1)^2 + \sum_{i=2}^{n} x_i^2\right]^{\frac{1}{2}} - \left[(4n - y_1)^2 + \sum_{i=2}^{n} y_i^2\right]^{\frac{1}{2}} \\ &= \frac{8n(y_1 - x_1) + \sum_{i=1}^{n}(x_i + y_i)(x_i - y_i)}{[(4n - x_1)^2 + \sum_{i=2}^{n} x_i^2]^{\frac{1}{2}} + [(4n - y_1)^2 + \sum_{i=2}^{n} y_i^2]^{\frac{1}{2}}} \\ &\geq \frac{8n(y_1 - x_1) - 4n(y_1 - x_1)}{16n} \geq \frac{y_1 - x_1}{4} \geq \frac{|x - y|}{4\sqrt{n}}.\end{aligned}$$

This shows that the above claim holds. For any $i \in \{1, \ldots, n\}$, let $f_i \in C_{\mathrm{c}}^{\infty} \subset \dot{W}^{s,X}$ satisfying $f_i(x) = |x - 4n\vec{e_i}|$ in $B(\mathbf{0}, 2)$. By (4.4) and the definition of f_i, we conclude that

$$\begin{aligned}&\left\|\left\|\frac{\mathbf{1}_{B(y,1)}(x)}{|x - y|^{s-\alpha-1}}\right\|_{X(x)} \mathbf{1}_{B(\mathbf{0},1)}(y)\right\|_{X(y)} \\ &\lesssim \left\|\left\|\frac{\mathbf{1}_{B(y,1)}(x) \sum_{i=1}^{n} \left||x - 4n\vec{e_i}| - |y - 4n\vec{e_i}|\right|}{|x - y|^{s-\alpha}}\right\|_{X(x)} \mathbf{1}_{B(\mathbf{0},1)}(y)\right\|_{X(y)} \\ &\leq \sum_{i=1}^{n} \|f_i\|_{\dot{W}^{s,X}} < \infty.\end{aligned}$$

This finishes the proof of the necessity and hence the proof of Lemma 4.3. □

The following corollary is a direct consequence of Lemma 4.3.

Corollary 4.4. *Let X, α, and s satisfy Assumption III(iii). Then* $C_{\mathrm{c}}^{\infty} \subset \dot{W}^{s,X}$.

Lemma 4.5. *Let X be a ball Banach function space satisfying Assumption III(i) and let* $f \in \mathscr{M}(\mathbb{R}^n \times \mathbb{R}^n)$ *satisfy* $\|\|f(x, y)\|_{X(x)}\|_{X(y)} < \infty$. *Then there exists a sequence of functions,* $\{f_m\}_{m\in\mathbb{N}} \subset C_{\mathrm{c}}(\mathbb{R}^n \times \mathbb{R}^n)$*, such that*

$$\lim_{m\to\infty} \left\| \|f_m(x, y) - f(x, y)\|_{X(x)}\right\|_{X(y)} = 0.$$

Proof. From Assumption III(i), we deduce that

$$\left\|\left\|f(x, y)\mathbf{1}_{B(\mathbf{0},j)^\complement}(x)\mathbf{1}_{B(\mathbf{0},j)^\complement}(y)\right\|_{X(x)}\right\|_{X(y)} \leq \left\|\left\|f(x, y)\mathbf{1}_{B(\mathbf{0},j)^\complement}(y)\right\|_{X(x)}\right\|_{X(y)} \to 0$$

as $j \to \infty$. Notice that, for almost every $y \in \mathbb{R}^n$, $\{x \in \mathbb{R}^n : f(x, y) > N\}$ converges to a set of zero Lebesgue measure as $N \to \infty$. By this and Assumption III(i), we find that, for almost every $y \in \mathbb{R}^n$,

$$\left\|f(x, y)\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x,y)>N\}}\right\|_{X(x)} \to 0$$

as $N \to \infty$, which, together with [49, Definition 3.11 and Proposition 3.12], further implies that

$$\left\|\left\|f(x, y)\mathbf{1}_{\{(x,y):\ f(x,y)>N\}}\right\|_{X(x)}\right\|_{X(y)} = \left\|\left\|f(x, y)\mathbf{1}_{\{x\in\mathbb{R}^n:\ f(x,y)>N\}}\right\|_{X(x)}\right\|_{X(y)} \to 0$$

as $N \to \infty$. Combining the above observations, we conclude that, for any $\varepsilon \in (0,\infty)$, there exists a bounded function g supported in a compact subset $D \subset (\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\left\| \|g(x,y) - f(x,y)\|_{X(x)} \right\|_{X(y)} < \varepsilon. \tag{4.5}$$

Using Lusin's theorem, we conclude that there exists a sequence of bounded continuous functions, $\{h_k(x,y)\}_{k\in\mathbb{N}}$, supported in D such that

$$\lim_{k\to\infty} [g(x,y) - h_k(x,y)] = 0$$

almost everywhere on $\mathbb{R}^n \times \mathbb{R}^n$. From Assumption III(i) and [42, Lemma 5.6.14], we deduce that, for almost every $y \in \mathbb{R}^n$,

$$\lim_{k\to\infty} \|g(x,y) - h_k(x,y)\|_{X(x)} = 0.$$

Then, by this, Assumption III(i), and [49, Definition 3.11 and Proposition 3.12], we find that

$$\lim_{k\to\infty} \left\| \|g(x,y) - h_k(x,y)\|_{X(x)} \right\|_{X(y)} = 0. \tag{4.6}$$

From (4.5) and (4.6), we conclude that, for any $\varepsilon \in (0,\infty)$, there exists a bounded continuous functions $h(x,y)$ supported in a set of finite measure $D \subset (\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\left\| \|h(x,y) - f(x,y)\|_{X(x)} \right\|_{X(y)} < \varepsilon.$$

This finishes the proof of Lemma 4.5. □

Lemma 4.6. *Let X and α satisfy Assumption I, $s \in (0, \min\{-\alpha, 1\})$, and X also satisfy Assumptions II and III. Let $u \in \mathscr{M}$ satisfy $\|u\|_{\dot{W}^{s,X}} < \infty$. Then there exists a sequence of functions, $\{u_m\}_{m\in\mathbb{N}} \subset C^\infty$, such that*

$$\lim_{m\to\infty} \|u - u_m\|_{\dot{W}^{s,X}} = 0.$$

Proof. Let $\rho \in C_{\mathrm{c}}^\infty$ be such that $\operatorname{supp}(\rho) \subset B(\mathbf{0},1)$ and $\int_{\mathbb{R}^n} \rho(x)dx = 1$, and define $\rho_m(\cdot) := m^n \rho(m\cdot)$ for any $m \in \mathbb{N}$. For any $x, y \in \mathbb{R}^n$, let

$$f(x,y) := \begin{cases} \dfrac{u(x) - u(y)}{|x-y|^{s-\alpha}}, & x \neq y, \\ 0, & x = y. \end{cases}$$

By Lemma 4.5, we find that, for any $\varepsilon \in (0,\infty)$, there exists $g \in C_{\mathrm{c}}(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\left\| \|f(x,y) - g(x,y)\|_{X(x)} \right\|_{X(y)} < \varepsilon. \tag{4.7}$$

From the definition of $\|\cdot\|_{\dot{W}^{s,X}}$, we infer that, for any $u \in \dot{W}^{s,X}$,

(4.8) $\|u * \rho_m - u\|_{\dot{W}^{s,X}}$

$$
\begin{aligned}
&= \left\| \left\| \frac{u*\rho_m(x) - u*\rho_m(y) - [u(x)-u(y)]}{|x-y|^{s-\alpha}} \right\|_{X(x)} \right\|_{X(y)} \\
&= \left\| \left\| \int_{\mathbb{R}^n} \frac{u(x-z)-u(y-z)}{|x-y|^{s-\alpha}} \rho_m(z)\,dz - \frac{u(x)-u(y)}{|x-y|^{s-\alpha}} \right\|_{X(x)} \right\|_{X(y)} \\
&= \left\| \left\| \int_{\mathbb{R}^n} f(x-z,y-z)\rho_m(z)\,dz - f(x,y) \right\|_{X(x)} \right\|_{X(y)} \\
&\le \left\| \|f(x,y)-g(x,y)\|_{X(x)} \right\|_{X(y)} \\
&\quad + \left\| \left\| \int_{\mathbb{R}^n} g(x-z,y-z)\rho_m(z)\,dz - g(x,y) \right\|_{X(x)} \right\|_{X(y)} \\
&\quad + \left\| \left\| \int_{\mathbb{R}^n} |f(x-z,y-z)-g(x-z,y-z)|\,\rho_m(z)\,dz \right\|_{X(x)} \right\|_{X(y)} \\
&=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}
$$

Notice that g is uniformly continuous and there exists $R \in (0,\infty)$ such that $\operatorname{supp}(g) \subset B(\mathbf{0},R) \times B(\mathbf{0},R)$. This, combined with the definition of ρ_m, further implies that

$$
\begin{aligned}
\mathrm{I}_2 &\le \sup_{\substack{(x,y)\in B(\mathbf{0},R+1)\times B(\mathbf{0},R+1)\\ (z,w)\in B(\mathbf{0},m^{-1})\times B(\mathbf{0},m^{-1})}} |g(x,y)-g(x-z,y-w)| \\
&\quad \times \left\| \left\| \mathbf{1}_{B(\mathbf{0},R+1)}(x)\mathbf{1}_{B(\mathbf{0},R+1)}(y) \right\|_{X(x)} \right\|_{X(y)} \\
&\to 0
\end{aligned} \tag{4.9}
$$

as $m \to \infty$. Now, we estimate I_3. Using Assumption III(ii), we conclude that

$$
\begin{aligned}
\mathrm{I}_3 &\lesssim \left\| \left\| \fint_{B(\mathbf{0},m^{-1})} |(f-g)(x-z,y-z)|\,dz \right\|_{X(x)} \right\|_{X(y)} \\
&\lesssim \left\| \|(f-g)(x,y)\|_{X(x)} \right\|_{X(y)} < \varepsilon.
\end{aligned} \tag{4.10}
$$

Combining (4.8), (4.7), (4.9), and (4.10), we find that $\|u*\rho_m - u\|_{\dot{W}^{s,X}} \to 0$ as $m \to \infty$, which completes the proof Lemma 4.6. □

The following Lorentz–Luxembourg lemma can be found in [80, Lemma 2.6].

Lemma 4.7. *Let X be a ball Banach function space. Then X coincides with its second associate space X''. In other words, a function f belongs to X if and only if it belongs to X'' and, in that case, $\|f\|_X = \|f\|_{X''}$.*

Lemma 4.8. *Let X be a ball Banach function space and $p, p' \in (1,\infty)$ satisfy $\frac{1}{p} + \frac{1}{p'} = 1$. Then, for any $f \in X^p$ and $g \in X^{p'}$, $\|fg\|_X \le \|f\|_{X^p}\|g\|_{X^{p'}}$.*

Proof. By Lemma 4.7, the definitions of X and X', Hölder's inequality, and the definitions of X^p and $X^{p'}$, we conclude that, for any $f \in X^p$ and $g \in X^{p'}$,

$$\begin{aligned}\|fg\|_X &= \sup_{\|h\|_{X'}=1} \int_{\mathbb{R}^n} |f(x)g(x)h(x)|\,dx \\ &\le \sup_{\|h\|_{X'}=1} \left[\int_{\mathbb{R}^n} |f(x)|^p\,|h(x)|\,dx\right]^{\frac{1}{p}} \left[\int_{\mathbb{R}^n} |f(x)|^{p'}\,|h(x)|\,dx\right]^{\frac{1}{p'}} \\ &\le \sup_{\|h\|_{X'}=1} \left[\int_{\mathbb{R}^n} |f(x)|^p\,|h(x)|\,dx\right]^{\frac{1}{p}} \sup_{\|h\|_{X'}=1} \left[\int_{\mathbb{R}^n} |f(x)|^{p'}\,|h(x)|\,dx\right]^{\frac{1}{p'}} \\ &= \|f\|_{X^p}\|g\|_{X^{p'}}.\end{aligned}$$

This finishes the proof of Lemma 4.8. □

Next, we prove Theorem 4.1.

Proof of Theorem 4.1. Let $[u] \in \mathcal{D}^{s,X}$. By the definition of $\mathcal{D}^{s,X}$, we find that there exists $\{u_m\}_{m\in\mathbf{N}} \subset C_{\mathrm{c}}^\infty$ converging to u with respect to the quasi-norm $\|\cdot\|_{\dot{W}^{s,X}}$. Using Assumption III(iii) and Corollary 4.4, we obtain $\{u_m\}_{m\in\mathbf{N}} \subset \dot{W}^{s,X}$ and hence $u \in \dot{W}^{s,X}$. By Corollary 3.5, we conclude that $\{u_m\}_{m\in\mathbf{N}} \subset X^{\frac{\alpha}{\alpha+s}}$. From Lemma 3.7, we infer that there exists a constant $\widetilde{C} \in \mathbb{C}$ such that, for any $\varepsilon \in (0,\infty)$, $\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |u(x)-\widetilde{C}|>\varepsilon\}}\|_X < \infty$. Let $i \in \mathbb{N}$. Using $\{u_m\}_{m\in\mathbf{N}} \subset C_{\mathrm{c}}^\infty$, we find that $\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |u(x)-\widetilde{C}-u_i(x)|>\varepsilon\}}\|_X < \infty$ and hence $u - \widetilde{C} - u_i \in \mathscr{M}_X$. From Theorem 3.2, we deduce that

$$\left\|u - \widetilde{C} - u_i\right\|_{X^{\frac{\alpha}{\alpha+s}}} \le \left\|u - \widetilde{C} - u_i\right\|_{\dot{W}^{s,X}},$$

which, together with $u_i \in X^{\frac{\alpha}{\alpha+s}}$, further implies that $u - \widetilde{C} \in X^{\frac{\alpha}{\alpha+s}}$. We then define

$$\mathcal{I}([u]) := u - \widetilde{C}.$$

Using the definition of $[u]$, we conclude that $\mathcal{I}$ is injective.

Now, we show that $\mathcal{I}$ is surjective. Let $u \in \dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$ and $g \in \mathscr{M}$ satisfy that $g \equiv 1$ on $B(\mathbf{0},1)$, $g \equiv 0$ on $B(\mathbf{0},2)^{\complement}$, and $g(x) := 2-|x|$ for any $x \in B(\mathbf{0},2)\setminus B(\mathbf{0},1)$. Let $g_j(\cdot) := g(\frac{\cdot}{j})$ for any $j \in \mathbb{N}$. Next, we prove that

$$\lim_{j\to\infty} \|u - g_j u\|_{\dot{W}^{s,X}} = 0. \tag{4.11}$$

It is easy to show that, for any $j \in \mathbb{N}$ and $x, y \in \mathbb{R}^n$,

$$|[1-g_j(x)]u(x) - [1-g_j(y)]u(y)| \le |u(x)-u(y)||1-g_j(x)| + |g_j(x)-g_j(y)||u(y)|$$

and hence

$$\begin{aligned}&\|u - g_j u\|_{\dot{W}^{s,X}} \\ &\quad\le \left\|\left\|\frac{|u(x)-u(y)||1-g_j(x)|}{|x-y|^{s-\alpha}}\right\|_{X(x)}\right\|_{X(y)} + \left\|\left\|\frac{|g_j(x)-g_j(y)||u(y)|}{|x-y|^{s-\alpha}}\right\|_{X(x)}\right\|_{X(y)} \\ &\quad=: I_j + J_j.\end{aligned} \tag{4.12}$$

We first estimate I_j. Form $\|u\|_{\dot{W}^{s,X}} < \infty$, we deduce that, for almost every $y \in \mathbb{R}^n$, $\|\frac{|u(\cdot)-u(y)|}{|\cdot-y|^{s-\alpha}}\|_X < \infty$. By this, the definition of g_j, and Assumption III(i), we conclude that, for almost every $y \in \mathbb{R}^n$,

$$\left\| \frac{|u(\cdot) - u(y)||1 - g_j(\cdot)|}{|\cdot - y|^{s-\alpha}} \right\|_X \leq \left\| \frac{|u(\cdot) - u(y)| \mathbf{1}_{B(\mathbf{0},j)^{\complement}}(\cdot)}{|\cdot - y|^{s-\alpha}} \right\|_X \to 0$$

as $j \to \infty$. Using this and [49, Definition 3.11 and Proposition 3.12], we obtain

$$\lim_{j\to\infty} I_j = 0. \tag{4.13}$$

This is the desired estimate for I_j.

Now, we estimate J_j. For any $j \in \mathbb{N}$ and $y \in \mathbb{R}^n$, let

$$f_j(y) := \left\| \frac{|g_j(\cdot) - g_j(y)|}{|\cdot - y|^{s-\alpha}} \right\|_X .$$

Using the definition of g_j and Assumption I, we conclude that

$$f_j(y) = j^{\alpha-s} \left\| \frac{|g(\frac{\cdot}{j}) - g(\frac{y}{j})|}{|\frac{\cdot}{j} - \frac{y}{j}|^{s-\alpha}} \right\|_X = j^{-s} \left\| \frac{|g(\cdot) - g(\frac{y}{j})|}{|\cdot - \frac{y}{j}|^{s-\alpha}} \right\|_X =: j^{-s} f\left(\frac{y}{j}\right).$$

Next, we claim that, for any $\beta \in (1, \infty)$, $f = f_1 \in X^\beta$. By the definition of g, we find that, for any $y \in B(\mathbf{0}, 3)^{\complement}$,

$$f(y) = \left\| \frac{|g(\cdot)|}{|\cdot - y|^{s-\alpha}} \right\|_X \lesssim \frac{\|\mathbf{1}_{B(\mathbf{0},2)}\|_X}{y^{s-\alpha}}. \tag{4.14}$$

From the definition of g, Assumptions I and III(iii), and $s \in (0, \min\{-\alpha, 1\})$, we deduce that, for any $y \in B(\mathbf{0}, 3)$,

$$\begin{aligned} f(y) &\leq \left\| \frac{|g(\cdot) - g(y)| \mathbf{1}_{B(\mathbf{0},4)}(\cdot)}{|\cdot - y|^{s-\alpha}} \right\|_X + \left\| \frac{|g(\cdot) - g(y)| \mathbf{1}_{B(\mathbf{0},4)^{\complement}}(\cdot)}{|\cdot - y|^{s-\alpha}} \right\|_X \\ &\lesssim \left\| \frac{\mathbf{1}_{B(y,7)}(\cdot)}{|\cdot - y|^{s-\alpha-1}} \right\|_X + \left\| \frac{\mathbf{1}_{B(\mathbf{0},4)^{\complement}}(\cdot)}{|\cdot|^{s-\alpha}} \right\|_X \\ &\leq 7^{-\alpha} \left\| \frac{\mathbf{1}_{B(y,7)}(7\cdot)}{|7\cdot - y|^{s-\alpha-1}} \right\|_X + \sum_{k=1}^{\infty} \left\| \frac{\mathbf{1}_{B(\mathbf{0},2^{k+2}) \setminus B(\mathbf{0},2^{k+1})}(\cdot)}{|\cdot|^{s-\alpha}} \right\|_X \\ &\leq 7^{1-s} \left\| \frac{\mathbf{1}_{B(\frac{y}{7},1)}(\cdot)}{|\cdot - \frac{y}{7}|^{s-\alpha-1}} \right\|_X + \sum_{k=1}^{\infty} 2^{-s(k+1)} \left\| \mathbf{1}_{B(\mathbf{0},2) \setminus B(\mathbf{0},1)} \right\|_X \lesssim 1. \end{aligned} \tag{4.15}$$

Combining (4.14), (4.15), Assumption III(iii), and $s \in (0, \min\{-\alpha, 1\})$, we conclude that, for any $\beta \in (1, \infty)$,

$$\begin{aligned} \|f\|_{X^\beta} &\leq \left\| f \mathbf{1}_{B(\mathbf{0},3)} \right\|_{X^\beta} + \left\| f \mathbf{1}_{B(\mathbf{0},3)^{\complement}} \right\|_{X^\beta} \\ &\lesssim \left\| \mathbf{1}_{B(\mathbf{0},3)} \right\|_{X^\beta} + \left\| \mathbf{1}_{B(\mathbf{0},2)} \right\|_X \left\| \frac{\mathbf{1}_{B(\mathbf{0},3)^{\complement}}(\cdot)}{|\cdot|^{\beta(s-\alpha)}} \right\|_X^{\frac{1}{\beta}} \\ &\leq \left\| \mathbf{1}_{B(\mathbf{0},3)} \right\|_{X^\beta} + \left\| \mathbf{1}_{B(\mathbf{0},2)} \right\|_X \left\{ \sum_{k=1}^{\infty} 3^{[-\beta s + (\beta-1)\alpha]k} \left\| \mathbf{1}_{B(\mathbf{0},3) \setminus B(\mathbf{0},1)} \right\|_X \right\}^{\frac{1}{\beta}} < \infty. \end{aligned}$$

This proves the above claim.

Let $u_j(\cdot) := j^{-s-\alpha}u(j\cdot)$ for any $j \in \mathbb{N}$. From the above claim and Assumption III(i), we infer that, for any $\varepsilon \in (0,\infty)$, there exists $\delta \in (0,\infty)$ such that

$$\left\|f\mathbf{1}_{B(\mathbf{0},\delta)}\right\|_{X^{-\frac{\alpha}{s}}} < \frac{\varepsilon}{2\|u\|_{X^{\frac{\alpha}{\alpha+s}}}}.$$

Using this, Lemma 4.8, and Assumption I, we conclude that, for any $j \in \mathbb{N}$,

$$\left\|fu_j\mathbf{1}_{B(\mathbf{0},\delta)}\right\|_X \le \left\|f\mathbf{1}_{B(\mathbf{0},\delta)}\right\|_{X^{-\frac{\alpha}{s}}} \left\|u_j\right\|_{X^{\frac{\alpha}{\alpha+s}}} = \left\|f\mathbf{1}_{B(\mathbf{0},\delta)}\right\|_{X^{-\frac{\alpha}{s}}} \|u\|_{X^{\frac{\alpha}{\alpha+s}}} < \frac{\varepsilon}{2}. \tag{4.16}$$

By Assumption III(i), we conclude that there exists $N \in \mathbb{N}$ such that, for any $j > N$,

$$\left\|u\mathbf{1}_{B(\mathbf{0},\delta j)^\complement}\right\|_{X^{\frac{\alpha}{\alpha+s}}} < \frac{\varepsilon}{2\|f\|_{X^{-\frac{\alpha}{s}}}}.$$

From this, Lemma 4.8, and Assumption I, we deduce that, for any $j > N$,

$$\left\|fu_j\mathbf{1}_{B(\mathbf{0},\delta)^\complement}\right\|_X \le \|f\|_{X^{-\frac{\alpha}{s}}} \left\|u_j\mathbf{1}_{B(\mathbf{0},\delta)^\complement}\right\|_{X^{\frac{\alpha}{\alpha+s}}} = \|f\|_{X^{-\frac{\alpha}{s}}} \left\|u\mathbf{1}_{B(\mathbf{0},\delta j)^\complement}\right\|_{X^{\frac{\alpha}{\alpha+s}}} < \frac{\varepsilon}{2}. \tag{4.17}$$

Using Assumption I, (4.16), and (4.17), we find that, for any $j > N$,

$$\left\|f_j u\right\|_X = \left\|fu_j\right\|_X \le \left\|fu_j\mathbf{1}_{B(\mathbf{0},\delta)^\complement}\right\|_X + \left\|fu_j\mathbf{1}_{B(\mathbf{0},\delta)}\right\|_X < \varepsilon. \tag{4.18}$$

This is the desired estimate for J_j. Then, combining (4.12), (4.13), and (4.18), we conclude that (4.11) holds. From this and Lemma 4.6, we infer that, for any $u \in \dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$, there exists a set $\{u_j\}_{j\in\mathbb{N}} \subset C_{\mathrm{c}}^\infty$ such that

$$\lim_{j\to\infty} \|u - u_j\|_{\dot{W}^{s,X}} = 0$$

and hence $[u] \in \mathcal{D}^{s,X}$, which further implies that $\mathcal{I}$ is surjective. This finishes the proof of Theorem 4.1. □

Now, we show that Assumption III(iii) is necessary for ball Banach function spaces whose quasi-norm is invariant under rotations in some weak sense.

Proposition 4.9. *Let X and α satisfy Assumption I and let $s \in (0, \min\{-\alpha, 1\})$. Assume that there exists a positive constant C such that, for any $n \times n$ unitary matrix A and any $f \in X$,*

$$\frac{1}{C}\|f\|_X \le \|f(A\cdot)\|_X \le C\|f\|_X. \tag{4.19}$$

If X has the property

$$\left\| \left\| \frac{\mathbf{1}_{B(y,1)}(x)}{|x-y|^{s-\alpha-1}} \right\|_{X(x)} \mathbf{1}_{B(\mathbf{0},1)}(y) \right\|_{X(y)} < \infty, \tag{4.20}$$

then X satisfies Assumption III(iii).

Proof. Assume that Assumption III(iii) fails. Then, for any $M \in (0,\infty)$, there exists $y_M \in B(\mathbf{0},1)$ such that

$$\left\| \frac{\mathbf{1}_{B(y_M,1)}(\cdot)}{|\cdot - y_M|^{s-\alpha-1}} \right\|_X \geq M. \tag{4.21}$$

By Assumption I and $s \in (0, \min\{-\alpha, 1\})$, we conclude that

$$\begin{aligned} \left\| \frac{\mathbf{1}_{B(\mathbf{0},2)}(\cdot)}{|\cdot|^{s-\alpha-1}} \right\|_X &\leq \sum_{i=0}^{\infty} 2^{(s-\alpha-1)i} \left\| \mathbf{1}_{B(\mathbf{0},2^{-i+1})\setminus B(\mathbf{0},2^{-i})} \right\|_X \\ &= \sum_{i=0}^{\infty} 2^{(s-1)i} \left\| \mathbf{1}_{B(\mathbf{0},2)\setminus B(\mathbf{0},1)} \right\|_X < \infty. \end{aligned}$$

From this, the obvious estimate that for any $x \in B(y_M, 2|y_M|)^\complement$ we have $|x - y_M| \sim |x|$, and (4.21), we obtain

$$\begin{aligned} \left\| \frac{\mathbf{1}_{B(y_M,2|y_M|)}(\cdot)}{|\cdot - y_M|^{s-\alpha-1}} \right\|_X &\geq \left\| \frac{\mathbf{1}_{B(y_M,1)}(\cdot)}{|\cdot - y_M|^{s-\alpha-1}} \right\|_X - \left\| \frac{\mathbf{1}_{B(y_M,1)\setminus B(y_M,2|y_M|)}(\cdot)}{|\cdot - y_M|^{s-\alpha-1}} \right\|_X \\ &\geq \left\| \frac{\mathbf{1}_{B(y_M,1)}(\cdot)}{|\cdot - y_M|^{s-\alpha-1}} \right\|_X - \left\| \frac{\mathbf{1}_{B(\mathbf{0},2)}(\cdot)}{|\cdot|^{s-\alpha-1}} \right\|_X \to \infty \end{aligned}$$

as $M \to \infty$. This fact together with Assumption I and (4.19) further implies that

$$\begin{aligned} &\left\| \left\| \frac{\mathbf{1}_{B(y,1)}(x)}{|x-y|^{s-\alpha-1}} \right\|_{X(x)} \mathbf{1}_{B(\mathbf{0},1)}(y) \right\|_{X(y)} \\ &\quad \geq \left\| \left\| \frac{\mathbf{1}_{B(y,2|y|)}(x)}{|x-y|^{s-\alpha-1}} \right\|_{X(x)} \mathbf{1}_{B(\mathbf{0},\frac{1}{2})}(y) \right\|_{X(y)} \\ &\quad = \left\| \left\| \frac{\mathbf{1}_{B(y,2|y|)}(\frac{|y|}{|y_M|}x)}{|\frac{|y|}{|y_M|}x - y|^{s-\alpha-1}} \right\|_{X(x)} \frac{|y|^{-\alpha}}{|y_M|^{-\alpha}} \mathbf{1}_{B(\mathbf{0},\frac{1}{2})}(y) \right\|_{X(y)} \\ &\quad = \left\| \frac{\mathbf{1}_{B(y_M,2|y_M|)}(\cdot)}{|\cdot - y_M|^{s-\alpha-1}} \right\|_X \left\| \frac{|\cdot|^{1-s}}{|y_M|^{1-s}} \mathbf{1}_{B(\mathbf{0},\frac{1}{2})}(\cdot) \right\|_X \to \infty \end{aligned}$$

as $M \to \infty$. This contradicts (4.20) and thus Assumption III(iii) must hold. This finishes the proof of Proposition 4.9. □

5. Applications to specific function spaces

In this section, we verify that our main results are applicable to several important examples of ball Banach function spaces, including Morrey spaces (Subsection 5.1), mixed-norm Lebesgue spaces (Subsection 5.2), Lebesgue spaces with power weights (Subsection 5.3), Besov–Triebel–Lizorkin–Bourgain–Morrey spaces (Subsection 5.4), and Lorentz spaces (Subsection 5.5). To the best of our knowledge, all results in this section are new. These applications reveal the extent to which Sobolev embeddings play a prominent role in function space theory. And we are certain that many other function spaces fall under the scope of our results.

To verify that these spaces satisfy some desired assumptions, we need the following lemma.

Lemma 5.1. *Let X and α satisfy Assumption I and let $s \in (0, \min\{-\alpha, 1\})$. Assume moreover that X satisfies the following property: there exists a positive constant C such that, for any $f \in X$ and $t \in \mathbb{R}^n$,*

$$\frac{1}{C} \|f(\cdot + t)\|_X \leq \|f\|_X \leq C \|f(\cdot + t)\|_X . \tag{5.1}$$

Then Assumption II and both (ii) and (iii) of Assumption III hold.

Proof. By (5.1), we find that, for any $x \in \mathbb{R}^n$, $\|\mathbf{1}_{B(x,1)}\|_X \sim \|\mathbf{1}_{B(\mathbf{0},1)}\|_X$ and hence Assumption II holds. From Minkowski's inequality and (5.1), we deduce that

$$\begin{aligned}\left\|\left\|\fint_{B(\mathbf{0},r)} f(x-z, y-z)\,dz\right\|_{X(x)}\right\|_{X(y)} &\lesssim \fint_{B(\mathbf{0},r)} \left\|\|f(x-z, y-z)\|_{X(x)}\right\|_{X(y)}\,dz \\ &\sim \left\|\|f(x, y)\|_{X(x)}\right\|_{X(y)} .\end{aligned}$$

This implies that Assumption III(ii) holds. Moreover, using (5.1), Assumption I, and $s \in (0, \min\{-\alpha, 1\})$, we conclude that, for any $y \in B(\mathbf{0}, 1)$,

$$\begin{aligned}\left\|\frac{\mathbf{1}_{B(y,1)}(\cdot)}{|\cdot - y|^{s-\alpha-1}}\right\|_X &\sim \left\|\frac{\mathbf{1}_{B(\mathbf{0},1)}(\cdot)}{|\cdot|^{s-\alpha-1}}\right\|_X \leq \sum_{k=1}^{\infty} 2^{(s-\alpha-1)k} \left\|\mathbf{1}_{B(\mathbf{0},2^{-k+1})\setminus B(\mathbf{0},2^{-k})}\right\|_X \\ &= \sum_{k=1}^{\infty} 2^{(s-1)k} \left\|\mathbf{1}_{B(\mathbf{0},2)\setminus B(\mathbf{0},1)}\right\|_X = \frac{2^{s-1}}{1-2^{s-1}} \left\|\mathbf{1}_{B(\mathbf{0},2)\setminus B(\mathbf{0},1)}\right\|_X = C' < \infty.\end{aligned}$$

This implies that Assumption III(iii) holds, which then completes the proof of Lemma 5.1. □

5.1. **Morrey Spaces.**

Recall that the Morrey space M_r^p with $0 < r \leq p < \infty$ was introduced by Morrey [60] in order to study the regularity of solutions to certain equations. Morrey spaces have many applications in the theory of elliptic partial differential equations, potential theory, and harmonic analysis; we refer to [13, 29, 30, 31, 33, 75] and the monographs [1, 66, 67, 78].

Definition 5.2. Let $0 < r \leq p < \infty$. The *Morrey space* M_r^p is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{M_r^p} := \sup_{B\in\mathbb{B}} |B|^{\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(B)} < \infty.$$

The following Sobolev-type embedding is a corollary of Theorem 3.2.

Theorem 5.3. *Let $0 < r \leq p < \infty$ and $s \in (0, \min\{\frac{n}{p}, 1\})$. Then there exists a positive constant C such that, for any $f \in \dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$ with $X := M_r^p$ and $\alpha := -\frac{n}{p}$,*

$$\sup_{B\in\mathbb{B}} \left[|B|^{\frac{r}{p}-1} \int_B |f(x)|^{\frac{rn}{n-sp}}\,dx\right]^{\frac{n-sp}{rn}}$$

$$\leq C \sup_{B_1,B_2\in\mathbb{B}} (|B_1||B_2|)^{\frac{1}{p}-\frac{1}{r}} \left\{\int_{B_1}\int_{B_2}\left[\frac{|f(x)-f(y)|}{|x-y|^{s+\frac{n}{p}}}\right]^r dx\,dy\right\}^{\frac{1}{r}}.$$

Proof. From the conclusion in [68, p. 87], we infer that M_r^p is a ball Banach function space. It is easy to show that Assumption I holds with X and α replaced, respectively, by M_r^p and $-\frac{n}{p}$. By the definition of M_r^p, we find that, for any $x \in \mathbb{R}^n$, $\|\mathbf{1}_{B(x,1)}\|_{M_r^p} = |B(\mathbf{0},1)|^{\frac{1}{p}}$, which implies that Assumption II holds with $X := M_r^p$. Thus, all the assumptions of Theorem 3.2 with $X := M_r^p$ and $\alpha := -\frac{n}{p}$ are satisfied. Then, using Theorem 3.2 with $X := M_r^p$ and $\alpha := -\frac{n}{p}$, we obtain the desired conclusions, thereby completing the proof of Theorem 5.3. □

Remark 5.4. From [72, Example 5.1], we know that the Morrey space M_r^p has no absolutely continuous norm if $1 < r < p < \infty$. Thus, it is still unknown whether or not Theorem 4.1 holds with $X := M_r^p$ and $\alpha := -\frac{n}{p}$.

5.2. **Mixed-Norm Lebesgue Spaces.** The mixed-norm Lebesgue space $L^{\vec{p}}$ was studied by Benedek and Panzone [6] in 1961, which can be traced back to Hörmander [37]. For more studies on mixed-norm Lebesgue spaces, we refer to [14, 15, 22, 23, 39, 40].

Definition 5.5. Let $\vec{p} := (p_1, \dots, p_n) \in (0,\infty]^n$. The *mixed-norm Lebesgue space* $L^{\vec{p}}$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{\vec{p}}} := \left\{\int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} |f(x_1,\dots,x_n)|^{p_1}\, dx_1\right]^{\frac{p_2}{p_1}} \cdots dx_n\right\}^{\frac{1}{p_n}} < \infty$$

with the usual modifications made when $p_i = \infty$ for some $i \in \{1,\dots,n\}$.

The following theorem is a corollary of Theorem 3.2.

Theorem 5.6. *Let* $\vec{p} := (p_1, \dots, p_n) \in (0,\infty]^n$ *and* $s \in (0, \min\{\sum_{i=1}^n \frac{1}{p_i}, 1\})$. *Then, for any* $f \in \dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$ *with* $X := L^{\vec{p}}$ *and* $\alpha := -\sum_{i=1}^n \frac{1}{p_i}$,

$$\left\{\int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} |f(x_1,\dots,x_n)|^{\frac{p_1\sum_{i=1}^n \frac{1}{p_i}}{\sum_{i=1}^n \frac{1}{p_i}-s}}\, dx_1\right]^{\frac{p_2}{p_1}} \cdots dx_n\right\}^{\frac{\sum_{i=1}^n \frac{1}{p_i}-s}{p_n\sum_{i=1}^n \frac{1}{p_i}}}$$

$$\lesssim \left\{\int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} \frac{|f(x)-f(y)|^{p_1}}{|x-y|^{(s+\sum_{i=1}^n \frac{1}{p_i})p_1}} dy_1\right]^{\frac{p_2}{p_1}} \cdots dy_n\right)^{\frac{p_1}{p_n}} dx_1\right]^{\frac{p_2}{p_1}} \cdots dx_n\right\}^{\frac{1}{p_n}}$$

with the implicit positive constant independent of f.

Proof. It is easy to prove that $L^{\vec{p}}$ is a ball Banach function space and Assumption I holds with X and α replaced, respectively, by $L^{\vec{p}}$ and $-\sum_{i=1}^{n}\frac{1}{p_i}$. By these, Lemma 5.1, and the translation invariance of $L^{\vec{p}}$, we conclude that all the assumptions of Theorem 3.2 with $X := L^{\vec{p}}$ and $\alpha := -\sum_{i=1}^{n}\frac{1}{p_i}$ are satisfied. Then, using Theorem 3.2 with $X := L^{\vec{p}}$ and $\alpha := -\sum_{i=1}^{n}\frac{1}{p_i}$, we obtain the desired conclusions, thereby completing the proof of Theorem 5.6. □

The following theorem is a corollary of Theorem 4.1.

Theorem 5.7. *Let $\vec{p} := (p_1, \dots, p_n) \in (0,\infty)^n$ and $s \in (0, \min\{\sum_{i=1}^{n}\frac{1}{p_i}, 1\})$. Then Theorem 4.1 holds with $X := L^{\vec{p}}$ and $\alpha := -\sum_{i=1}^{n}\frac{1}{p_i}$.*

Proof. It is straightforward that $L^{\vec{p}}$ has an absolutely continuous norm. By this, the proof of Theorem 5.6, Lemma 5.1, and the translation invariance of $L^{\vec{p}}$, we conclude that all the assumptions of Theorem 4.1 with $X := L^{\vec{p}}$ and $\alpha := -\sum_{i=1}^{n}\frac{1}{p_i}$ are satisfied. Then, using Theorem 4.1 with $X := L^{\vec{p}}$ and $\alpha := -\sum_{i=1}^{n}\frac{1}{p_i}$, we obtain the desired conclusions, which then completes the proof of Theorem 5.7. □

5.3. **Lebesgue Spaces with Power Weights.** We first present the definitions of both Muckenhoupt weights and weighted Lebesgue spaces (see, for instance, [24, Definitions 7.1.2 and 7.1.3]).

Definition 5.8. Let $p \in [1,\infty)$ and ω be a nonnegative locally integrable function on $\mathbb{R}^n$. Then ω is called an A_p *weight*, denoted by $\omega \in A_p$, if, when $p \in (1,\infty)$,

$$[\omega]_{A_p} := \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B \omega(x)\,dx \left\{\frac{1}{|B|}\int_B [\omega(x)]^{-\frac{1}{p-1}}\,dx\right\}^{p-1} < \infty$$

and

$$[\omega]_{A_1} := \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B \omega(x)\,dx \left\{\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}[\omega(x)]^{-1}\right\} < \infty,$$

where the suprema are taken over all balls $B \in \mathbb{B}$. Moreover, the *class* A_∞ is defined by setting

$$A_\infty := \bigcup_{p\in[1,\infty)} A_p.$$

Definition 5.9. Let $p \in (0,\infty)$ and $\omega \in A_\infty$. The *weighted Lebesgue space* L^p_ω is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p_\omega} = \left[\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\,dx\right]^{\frac{1}{p}} < \infty.$$

The following theorem is a corollary of Theorem 3.2.

Theorem 5.10. *Let $p \in (1,\infty)$, $\omega(x) := |x|^\beta$ with $\beta \in (0, n(p-1))$ for any $x \in \mathbb{R}^n$, and $s \in (0, \min\{\frac{n+\beta}{p}, 1\})$. Then there exists a positive constant C such that, for any $f \in \dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$ with $X := L^p_\omega$ and $\alpha := -\frac{n+\beta}{p}$,*

$$\left[\int_{\mathbb{R}^n} |u(x)|^{\frac{p(n+\beta)}{n+\beta-sp}} |x|^\beta \, dx\right]^{\frac{n+\beta-sp}{n+\beta}} \leq C \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\beta+sp}} |x|^\beta |y|^\beta \, dx\, dy$$

Proof. From [20, p. 141], we know that $|x|^\beta \in A_p$. Combining this and [68, p. 86], we conclude that L^p_ω is a ball Banach function space. It is easy to prove that Assumption I holds with X and α replaced, respectively, by L^p_ω and $-\frac{n+\beta}{p}$. From $\beta \in (0, n(p-1))$, we infer that, for any $x \in \mathbb{R}^n$, $\|\mathbf{1}_{B(x,1)}\|_{L^p_\omega} \geq \|\mathbf{1}_{B(\mathbf{0},1)}\|_{L^p_\omega}$, which implies that Assumption II holds with $X := L^p_\omega$. Thus, all the assumptions of Theorem 3.2 with $X := L^p_\omega$ and $\alpha := -\frac{n+\beta}{p}$ are satisfied. Then, using Theorem 3.2 with $X := L^p_\omega$ and $\alpha := -\frac{n+\beta}{p}$, we obtain the desired conclusions and hence complete the proof of Theorem 5.10. □

The following theorem is a corollary of Theorem 4.1.

Theorem 5.11. *If $p \in (1,\infty)$, $\omega(x) := |x|^\beta$ for any $x \in \mathbb{R}^n$ and some $\beta \in (0, \min\{p, \frac{n(p-1)}{2}\})$, and $s \in (0, \min\{\frac{n+\beta}{p}, \frac{p-\beta}{p}, 1\})$, then Theorem 4.1 holds with $X := L^p_\omega$ and $\alpha := -\frac{n+\beta}{p}$.*

Proof. It is easy to show that L^p_ω has an absolutely continuous norm. By [20, p. 141], we find that, for any $t \in \mathbb{R}^n$, $|x+t|^{2\beta} \in A_p$. Using this together with [24, Exercise 7.1.9], we conclude that, for any $t \in \mathbb{R}^n$, $|x|^\beta|x+t|^\beta \in A_p$. Then, from [20, Theorem 7.3], we deduce that

$$\begin{aligned}
&\left\|\left\|\fint_{B(\mathbf{0},r)} f(x+z, y+z)\, dz\right\|_{L^p_\omega}\right\|_{L^p_\omega}\\
&= \left[\iint_{\mathbb{R}^n\times\mathbb{R}^n} \fint_{B(\mathbf{0},r)} f(x+z, y+z)^p |x|^\beta |y|^\beta \, dz\, dx\, dy\right]^{\frac{1}{p}}\\
&= \left[\iint_{\mathbb{R}^n\times\mathbb{R}^n} \fint_{B(\mathbf{0},r)} f(x+z, x+z+t)^p |x|^\beta |x+t|^\beta \, dz\, dx\, dt\right]^{\frac{1}{p}}\\
&\lesssim \left[\iint_{\mathbb{R}^n\times\mathbb{R}^n} f(x, x+t)^p |x|^\beta |x+t|^\beta \, dx\, dt\right]^{\frac{1}{p}}\\
&= \left[\iint_{\mathbb{R}^n\times\mathbb{R}^n} f(x, y)^p |x|^\beta |y|^\beta \, dx\, dy\right]^{\frac{1}{p}} = \left\|\|f(x,y)\|_{L^p_\omega}\right\|_{L^p_\omega}.
\end{aligned}$$

This proves that Assumption III(ii) holds with $X := L^p_\omega$. By the definition of $\|\cdot\|_{L^p_\omega}$ and $s \in (0, \min\{\frac{n+\beta}{p}, \frac{p-\beta}{p}, 1\})$, we conclude that, for any $y \in B(\mathbf{0}, 1)$,

$$\left\|\frac{\mathbf{1}_{B(y,1)}(\cdot)}{|\cdot - y|^{s-1+\frac{n+\beta}{p}}}\right\|_{L^p_\omega} = \int_{B(y,1)} \frac{|x|^\beta}{|x-y|^{(s-1)p+n+\beta}}\, dx \lesssim \int_{B(\mathbf{0},2)} \frac{1}{|x-y|^{(s-1)p+n+\beta}}\, dx < \infty.$$

This shows that Assumption III(iii) holds with $X := L^p_\omega$. Combining the above observations and the proof of Theorem 5.10, we conclude that all the assumptions of Theorem 4.1 with $X := L^p_\omega$ and $\alpha := -\frac{n+\beta}{p}$ are satisfied. Then, using Theorem 4.1 with $X := L^p_\omega$ and $\alpha := -\frac{n+\beta}{p}$, we obtain the desired conclusions, which completes the proof of Theorem 5.11. □

5.4. **Besov–Triebel–Lizorkin–Bourgain–Morrey Spaces.** It is well known that Morrey-type spaces, serving as a good substitute of Morrey spaces, have been found many applications in harmonic analysis and partial differential equations; see, for example, [21, 25, 43, 53, 74]. To study the Bochner–Riesz multiplier problems in $\mathbb{R}^3$, Bourgain [9] introduced a special Bourgain–Morrey spaces. Subsequently, Masaki [50] introduced Bourgain–Morrey spaces for the full range of exponents to explore some problems on nonlinear Schrödinger equations. In addition, Bourgain–Morrey spaces have many applications in the theory of partial differential equations (see, for example, [5, 10, 41, 51, 52, 61, 62]). Recently, Hatano et al. [35] revealed several fundamental real-variable properties of Bourgain–Morrey spaces. Motivated by Bourgain–Morrey spaces and the structure of Besov spaces (or Triebel–Lizorkin spaces), Zhao et al. [81] and Hu et al. [38] respectively introduced Besov–Bourgain–Morrey spaces and Triebel–Lizorkin–Bourgain–Morrey spaces as follows.

Definition 5.12. Let $0 < q \le p \le r \le \infty$, $\tau \in (0,\infty]$, and $\{Q_{v,m}\}_{v\in\mathbb{Z},\, m\in\mathbb{Z}_n}$ be the system of dyadic cubes of $\mathbb{R}^n$.

(i) The *Besov–Bourgain–Morrey space* $\dot{M}B^{p,\tau}_{q,r}$ is defined to be the set of all $f \in L^q_{\mathrm{loc}}$ such that

$$\|f\|_{\dot{M}B^{p,\tau}_{q,r}} := \left\{\sum_{v\in\mathbb{Z}}\left[\sum_{m\in\mathbb{Z}^n}\left(|Q_{vm}|^{\frac{1}{p}-\frac{1}{q}}\left\|f\mathbf{1}_{Q_{vm}}\right\|_{L^q}\right)^r\right]^{\frac{\tau}{r}}\right\}^{\frac{1}{\tau}} < \infty$$

with the usual modifications made when $r = \infty$ and $\tau = \infty$.

(ii) The *Triebel–Lizorkin–Bourgain–Morrey space* $\dot{M}F^{p,\tau}_{q,r}$ is defined to be the set of all $f \in L^q_{\mathrm{loc}}$ such that

$$\|f\|_{\dot{M}F^{p,\tau}_{q,r}} := \left(\int_{\mathbb{R}^n}\left\{\int_0^\infty\left[t^{n(\frac{1}{p}-\frac{1}{q}-\frac{1}{r})}\left\|f\mathbf{1}_{B(y,t)}\right\|_{L^q}\right]^\tau\frac{dt}{t}\right\}^{\frac{r}{\tau}}dy\right)^{\frac{1}{r}} < \infty$$

with the usual modifications made when $r = \infty$ and $\tau = \infty$.

The following theorem is a corollary of Theorem 3.2.

Theorem 5.13. *Let both* $0 \le q < p < r \le \infty$ *and* $\tau \in (1,\infty)$ *or* $1 \le q \le p \le r \le \tau = \infty$, $A \in \{B, F\}$, $s \in (0, \min\{\frac{n}{p}, 1\})$, *and* $\gamma := \frac{n}{n-sp}$. *Then there exists a positive* *constant C such that, for any $f \in \dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$ with $X := \dot{M}A^{p,\tau}_{q,r}$ and $\alpha := -\frac{n}{p}$,*

$$\|f\|_{\dot{M}A^{\gamma p,\gamma\tau}_{\gamma q,\gamma r}} \le C \left\| \left\| \frac{|f(x)-f(y)|}{|x-y|^{s+\frac{n}{p}}} \right\|_{\dot{M}A^{p,\tau}_{q,r}} \right\|_{\dot{M}A^{p,\tau}_{q,r}}.$$

Proof. We only consider the case $A = B$ because the proof of the case $A = F$ is similar and hence we omit the details. From the proof of [82, Lemma 4.10], we infer that $\dot{M}B^{p,\tau}_{q,r}$ is a ball Banach function space. It is easy to prove that Assumption I holds with X and α replaced, respectively, by $\dot{M}B^{p,\tau}_{q,r}$ and $-\frac{n}{p}$. By these, Lemma 5.1, and the translation invariance of $\dot{M}B^{p,\tau}_{q,r}$, we conclude that all the assumptions of Theorem 3.2 with $X := \dot{M}B^{p,\tau}_{q,r}$ and $\alpha := -\frac{n}{p}$ are satisfied. Then, using Theorem 3.2 with $X := \dot{M}B^{p,\tau}_{q,r}$ and $\alpha := -\frac{n}{p}$, we obtain the desired conclusions, which completes the proof of Theorem 5.13. □

The following theorem is a corollary of Theorem 4.1.

Theorem 5.14. *Let $0 < q < p < r < \infty$, $\tau \in (1,\infty)$, $A \in \{B, F\}$, and $s \in (0, \min\{\frac{n}{p}, 1\})$. Then Theorem 4.1 holds with $X := \dot{M}A^{p,\tau}_{q,r}$ and $\alpha := -\frac{n}{p}$.*

Proof. We only consider the case $A = B$ because the proof of the case $A = F$ is similar and hence we omit the details. From the proof of [82, Theorem 4.12], we infer that $\dot{M}B^{p,\tau}_{q,r}$ has an absolutely continuous norm. By this, the proof of Theorem 5.13, Lemma 5.1, and the translation invariance of $\dot{M}B^{p,\tau}_{q,r}$, we conclude that all the assumptions of Theorem 4.1 with $X := \dot{M}B^{p,\tau}_{q,r}$ and $\alpha := -\frac{n}{p}$ are satisfied. Then, using Theorem 4.1 with $X := \dot{M}B^{p,\tau}_{q,r}$ and $\alpha := -\frac{n}{p}$, we obtain the desired conclusions, which completes the proof of Theorem 5.14. □

5.5. **Lorentz Spaces.** The Lorentz space was studied by Lorentz [47, 48] in the early 1950's. As a natural generalization of Lebesgue spaces, Lorentz spaces serve as the intermediate spaces of Lebesgue spaces in the real interpolation (see, for example, [12]). For more studies on Lorentz spaces and their associated function spaces, we refer to [64, 73, 44, 45].

Definition 5.15. Let $p \in (0,\infty)$ and $q \in (0,\infty]$. For any $f \in \mathscr{M}$, let

$$a_f(\lambda) := |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|$$

and

$$f^*(t) := \inf\{\lambda \in (0,\infty) : a_f(\lambda) \le t\}.$$

The *Lorentz space* $L^{p,q}$ is defined to be the set of all functions $f \in \mathscr{M}$ such that, when $p, q \in (0,\infty)$,

$$\|f\|_{L^{p,q}} := \left\{ \frac{q}{p} \int_0^\infty \left[t^{\frac{1}{p}} f^*(t) \right]^q \frac{dt}{t} \right\}^{\frac{1}{q}} < \infty$$

and

$$\|f\|_{L^{p,\infty}} := \sup_{t\in(0,\infty)} t^{\frac{1}{p}} f^*(t) < \infty.$$

The following theorem is a corollary of Theorem 3.2.

Theorem 5.16. *Let* $p \in (1,\infty)$, $q \in (1,\infty]$, *and* $s \in (0, \min\{\frac{n}{p}, 1\})$. *Then there exists a positive constant C such that, for any* $f \in \dot{W}^{s,X} \cap X^{\frac{\alpha}{\alpha+s}}$ *with* $X := L^{p,q}$ *and* $\alpha := -\frac{n}{p}$,

$$\|f\|_{L^{\frac{p^2}{p-sn},\frac{pq}{p-sn}}} \le C \left\| \left\| \frac{|f(x) - f(y)|}{|x-y|^{s+\frac{n}{p}}} \right\|_{L^{p,q}} \right\|_{L^{p,q}}.$$

Proof. From [68, p. 87], we infer that $L^{p,q}$ is a ball Banach function space. It is easy to show that Assumption I holds with X and α replaced, respectively, by $L^{p,q}$ and $-\frac{n}{p}$. By these, Lemma 5.1, and the translation invariance of $L^{p,q}$, we conclude that all the assumptions of Theorem 3.2 with $X := L^{p,q}$ and $\alpha := -\frac{n}{p}$ are satisfied. Then, using Theorem 3.2 with $X := L^{p,q}$ and $\alpha := -\frac{n}{p}$, we obtain the desired conclusions, which completes the proof of Theorem 5.16. □

The following theorem is a corollary of Theorem 4.1.

Theorem 5.17. *Let* $p \in (1,\infty)$, $q \in (1,\infty)$, *and* $s \in (0, \min\{\frac{n}{p}, 1\})$. *Then Theorem 4.1 holds with* $X := L^{p,q}$ *and* $\alpha := -\frac{n}{p}$.

Proof. From [76, Remark 3.4(iii)], we infer that $L^{p,q}$ has an absolutely continuous norm. By this, the proof of Theorem 5.16, Lemma 5.1, and the translation invariance of $L^{p,q}$, we conclude that all the assumptions of Theorem 4.1 with $X := L^{p,q}$ and $\alpha := -\frac{n}{p}$ are satisfied. Then, using Theorem 4.1 with $X := L^{p,q}$ and $\alpha := -\frac{n}{p}$, we obtain the desired conclusions, which completes the proof of Theorem 5.17. □

Acknowledgements. Yiqun Chen would like to express his deep thanks to Professor Ziyi He for some useful advice on Lemma 3.4. The authors would also like to thank the referee for her/his careful reading and several valuable comments which improved the presentation of this article.

This project is supported by the National Key Research and Development Program of China (Grant No. 2020YFA0712900), the National Natural Science Foundation of China (Grant Nos. 12431006 and 12371093), and the Fundamental Research Funds for the Central Universities (Grant No. 2233300008). Loukas Grafakos is supported by a Simons Grant (Grant No. 624733).

References

[1] D. R. Adams, *Morrey Spaces*, Lect. Notes Appl. Numer. Harmon. Anal., Birkhäuser/Springer, Cham, 2015.

[2] R. Alvarado, P. Górka and P. Hajłasz, *Sobolev embedding for $M^{1,p}$ spaces is equivalent to a lower bound of the measure*, J. Funct. Anal. 279 (2020), Paper No. 108628, 39 pp.

[3] R. Alvarado, D. Yang and W. Yuan, *A measure characterization of embedding and extension domains for Sobolev, Triebel–Lizorkin, and Besov spaces on spaces of homogeneous type*, J. Funct. Anal. 283 (2022), Paper No. 109687, 71 pp.

[4] R. Alvarado, D. Yang and W. Yuan, *Optimal embeddings for Triebel–Lizorkin and Besov spaces on quasi-metric measure spaces*, Math. Z. 307 (2024), Paper No. 50, 59 pp.

[5] P. Bégout and A. Vargas, *Mass concentration phenomena for the L^2-critical nonlinear Schrödinger equation*, Trans. Amer. Math. Soc. 359 (2007), 5257–5282.

[6] A. Benedek and R. Panzone, *The space L^P, with mixed norm*, Duke Math. J. 28 (1961), 301–324.

[7] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure Appl. Math. 129, Academic Press, Boston, MA, 1988.

[8] B. Bojarski, *Remarks on some geometric properties of Sobolev mappings*, in: Functional Analysis & Related Topics (Sapporo, 1990), pp. 65–76, World Sci. Publ., River Edge, NJ, 1991.

[9] J. Bourgain, *On the restriction and multiplier problems in $\mathbb{R}^3$*, in: Geometric Aspects of Functional Analysis (1989–90), pp. 179–191, Lecture Notes in Math. 1469, Springer, Berlin, 1991.

[10] J. Bourgain, *Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity*, Internat. Math. Res. Notices 1998 (1998), 253–283.

[11] L. Brasco, D. Gómez-Castro and J. Vázquez, *Characterisation of homogeneous fractional Sobolev spaces*, Calc. Var. Partial Differential Equations 60 (2021), Paper No. 60, 40 pp.

[12] A.-P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. 24 (1964), 113–190.

[13] F. Chiarenza and M. Frasca, *Morrey spaces and Hardy–Littlewood maximal function*, Rend. Mat. Appl. (7) 7 (1987), 273–279 (1988).

[14] G. Cleanthous, A. G. Georgiadis and M. Nielsen, *Anisotropic mixed-norm Hardy spaces*, J. Geom. Anal. 27 (2017), 2758–2787.

[15] G. Cleanthous, A. G. Georgiadis and M. Nielsen, *Fourier multipliers on anisotropic mixed-norm spaces of distributions*, Math. Scand. 124 (2019), 289–304.

[16] F. Dai, L. Grafakos, Z. Pan, D. Yang, W. Yuan and Y. Zhang, *The Bourgain–Brezis–Mironescu formula on ball Banach function spaces*, Math. Ann. 388 (2024), 1691–1768.

[17] F. Dai, X. Lin, D. Yang, W. Yuan and Y. Zhang, *Poincaré inequality meets Brezis–Van Schaftingen–Yung formula on metric measure spaces*, J. Funct. Anal. 283 (2022), 109645, 52 pp.

[18] F. Dai, X. Lin, D. Yang, W. Yuan and Y. Zhang, *Brezis–Van Schaftingen–Yung formulae in ball Banach function spaces with applications to fractional Sobolev and Gagliardo–Nirenberg inequalities*, Calc. Var. Partial Differential Equations 62 (2023), Paper No. 56, 73 pp.

[19] R. Del Campo, A. Fernández, F. Mayoral and F. Naranjo, *Orlicz spaces associated to a Banach function space: Applications to vector measures and interpolation*, Collect. Math. 72 (2021), 481–499.

[20] J. Duoandikoetxea, *Fourier Analysis*, Graduate Studies in Mathematics 29, American Mathematical Society, Providence, RI, 2001.

[21] G. Di Fazio and T. Nguyen, *Regularity estimates in weighted Morrey spaces for quasilinear elliptic equations*, Rev. Mat. Iberoam. 36 (2020), 1627–1658.

[22] A. G. Georgiadis, J. Johnsen and M. Nielsen, *Wavelet transforms for homogeneous mixed-norm Triebel–Lizorkin spaces*, Monatsh. Math. 183 (2017), 587–624.

[23] A. G. Georgiadis and M. Nielsen, *Pseudodifferential operators on mixed-norm Besov and Triebel–Lizorkin spaces*, Math. Nachr. 289 (2016), 2019–2036.

[24] L. Grafakos, *Classical Fourier Analysis*, Third edition, Graduate Texts in Mathematic 249, Springer, New York, 2014.

[25] D. I. Hakim and Y. Sawano, *Complex interpolation of variable Morrey spaces*, Math. Nachr. 294 (2021), 2140–2150.

[26] D. D. Haroske, *Envelopes and Sharp Embeddings of Function Spaces*, Chapman & Hall/CRC Research Notes in Mathematics 437. Chapman & Hall/CRC, Boca Raton, FL, 2007.

[27] D. D. Haroske, *Sobolev spaces with Muckenhoupt weights, singularities and inequalities*, Georgian Math. J. 15 (2008), 263–280.

[28] D. D. Haroske, H. Leopold and L. Skrzypczak, *Nuclear embeddings in general vector-valued sequence spaces with an application to Sobolev embeddings of function spaces on quasi-bounded domains*, J. Complexity 69 (2022), Paper No. 101605, 23 pp.

[29] D. D. Haroske, S. D. Moura and L. Skrzypczak, *Some embeddings of Morrey spaces with critical smoothness*, J. Fourier Anal. Appl. 26 (2020), Paper No. 50, 31 pp.

[30] D. D. Haroske, S. D. Moura, C. Schneider and L. Skrzypczak, *Unboundedness properties of smoothness Morrey spaces of regular distributions on domains*, Sci. China Math. 60 (2017), 2349–2376.

[31] D. D. Haroske, C. Schneider and L. Skrzypczak, *Morrey spaces on domains: Different approaches and growth envelopes*, J. Geom. Anal. 28 (2018), 817–841.

[32] D. D. Haroske and L. Skrzypczak, *On Sobolev and Franke–Jawerth embeddings of smoothness Morrey spaces*, Rev. Mat. Complut. 27 (2014), 541–573.

[33] D. D. Haroske and L. Skrzypczak, *Embeddings of weighted Morrey spaces*, Math. Nachr. 290 (2017), 1066–1086.

[34] D. D. Haroske and H. Triebel, *Distributions, Sobolev Spaces, Elliptic Equations*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008.

[35] N. Hatano, T. Nogayama, Y. Sawano and D. I. Hakim, *Bourgain–Morrey spaces and their applications to boundedness of operators*, J. Funct. Anal. 284 (2023), Paper No. 109720, 52 pp.

[36] K.-P. Ho, *Sobolev–Jawerth embedding of Triebel–Lizorkin–Morrey–Lorentz spaces and fractional integral operator on Hardy type spaces*, Math. Nachr. 287 (2014), 1674–1686.

[37] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. 104 (1960), 93–140.

[38] P. Hu, Y. Li and D. Yang, *Bourgain–Morrey spaces meet structure of Triebel–Lizorkin spaces*, Math. Z. 304 (2023), Paper No. 19, 49 pp.

[39] L. Huang, J. Liu, D. Yang and W. Yuan, *Atomic and Littlewood–Paley characterizations of anisotropic mixed-norm Hardy spaces and their applications*, J. Geom. Anal. 29 (2019), 1991–2067.

[40] L. Huang, J. Liu, D. Yang and W. Yuan, *Dual spaces of anisotropic mixed-norm Hardy spaces*, Proc. Amer. Math. Soc. 147 (2019), 1201–1215.

[41] C. E. Kenig, G. Ponce and L. Vega, *On the concentration of blow up solutions for the generalized KdV equation critical in L^2*, in: Nonlinear Wave Equations (Providence, RI, 1998), pp. 131–156, Contemp. Math. 263, Amer. Math. Soc., Providence, RI, 2000.

[42] Y. Li, D. Yang and L. Huang, *Real-Variable Theory of Hardy Spaces Associated with Generalized Herz Spaces of Rafeiro and Samko*, Lecture Notes in Math. 2320, Springer, Singapore, 2022.

[43] C. Lin and Q. Yang, *Semigroup characterization of Besov type Morrey spaces and well-posedness of generalized Navier–Stokes equations*, J. Differential Equations 254 (2013), 804–846.

[44] J. Liu, F. Weisz, D. Yang and W. Yuan, *Littlewood–Paley and finite atomic characterizations of anisotropic variable Hardy–Lorentz spaces and their applications*, J. Fourier Anal. Appl. 25 (2019), 874–922.

[45] J. Liu, D. Yang and W. Yuan, *Anisotropic Hardy–Lorentz spaces and their applications*, Sci. China Math. 59 (2016), 1669–1720.

[46] L. Liu and J. Xiao, *Fractional Hardy-Sobolev L^1-embedding per capacity-duality*, Appl. Comput. Harmon. Anal. 51 (2021), 17–55.

[47] G. G. Lorentz, *Some new functional spaces*, Ann. of Math. (2) 51 (1950), 37–55.

[48] G. G. Lorentz, *On the theory of spaces* Λ, Pacific J. Math. 1 (1951), 411–429.

[49] E. Lorist and Z. Nieraeth, *Banach function spaces done right*, Indag. Math. (N.S.) 35 (2024), 247–268.

[50] S. Masaki, *Two minimization problems on non-scattering solutions to mass-subcritical non-linear Schrödinger equation*, arXiv: 1605.09234.

[51] S. Masaki and J. Segata, *Existence of a minimal non-scattering solution to the mass-subcritical generalized Korteweg–de Vries equation*, Ann. Inst. H. Poincar C Anal. Non Linéaire 35 (2018), 283–326.

[52] S. Masaki and J. Segata, *Refinement of Strichartz estimates for Airy equation in nondiagonal case and its application*, SIAM J. Math. Anal. 50 (2018), 2839–2866.

[53] M. Mastyło, Y. Sawano and H. Tanaka, *Morrey-type space and its Köthe dual space*, Bull. Malays. Math. Sci. Soc. 41 (2018), 1181–1198.

[54] V. Maz'ya, *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, Second, Revised and Augmented Edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 342, Springer, Heidelberg, 2011.

[55] Y. Mizuta, E. Nakai, T. Ohno and T. Shimomura, *An elementary proof of Sobolev embeddings for Riesz potentials of functions in Morrey spaces* $L^{1,\nu,\beta}(G)$, Hiroshima Math. J. 38 (2008), 425–436.

[56] Y. Mizuta, E. Nakai, T. Ohno and T. Shimomura, *Boundedness of fractional integral operators on Morrey spaces and Sobolev embeddings for generalized Riesz potentials*, J. Math. Soc. Japan 62 (2010), 707–744.

[57] Y. Mizuta, E. Nakai, T. Ohno and T. Shimomura, *Sobolev's inequality for Riesz potentials in Orlicz–Musielak spaces of variable exponent*, in: Banach and Function Spaces III (ISBFS 2009), pp. 409–419, Yokohama Publ., Yokohama, 2011.

[58] Y. Mizuta, E. Nakai, T. Ohno and T. Shimomura, *Riesz potentials and Sobolev embeddings on Morrey spaces of variable exponents*, Complex Var. Elliptic Equ. 56 (2011), 671–695.

[59] Y. Mizuta, E. Nakai, T. Ohno and T. Shimomura, *Maximal functions, Riesz potentials and Sobolev embeddings on Musielak–Orlicz–Morrey spaces of variable exponent in* $\mathbb{R}^n$, Rev. Mat. Complut. 25 (2012), 413–434.

[60] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. 43 (1938), 126–166.

[61] A. Moyua, A. Vargas and L. Vega, *Schrödinger maximal function and restriction properties of the Fourier transform*, Internat. Math. Res. Notices 1996 (1996), 793–815.

[62] A. Moyua, A. Vargas and L. Vega, *Restriction theorems and maximal operators related to oscillatory integrals in* $\mathbb{R}^3$, Duke Math. J. 96 (1999), 547–574.

[63] E. D. Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (2012), 521–573.

[64] R. Oberlin, A. Seeger, T. Tao, C. Thiele and J. Wright, *A variation norm Carleson theorem*, J. Eur. Math. Soc. (JEMS) 14 (2012), 421–464.

[65] J. Peetre, *Espaces d'interpolation et théorème de Soboleff*, Ann. Inst. Fourier (Grenoble) 16 (1966), fasc. 1, 279–317.

[66] Y. Sawano, G. Di Fazio and D. Hakim, *Morrey Spaces, Introduction and Applications to Integral Operators and PDE's*, Vol. I, Monogr. Res. Notes Math., CRC Press, Boca Raton, FL, 2020.

[67] Y. Sawano, G. Di Fazio and D. Hakim, *Morrey Spaces, Introduction and Applications to Integral Operators and PDE's*, Vol. II, Monogr. Res. Notes Math., CRC Press, Boca Raton, FL, 2020.

[68] Y. Sawano, K.-P. Ho, D. Yang and S. Yang, *Hardy spaces for ball Banach function spaces*, Dissertationes Math. (Rozprawy Mat.) 525 (2017), 1–102.

[69] Y. Sawano and T. Shimomura, *Sobolev embeddings for Riesz potentials of functions in non-doubling Morrey spaces of variable exponents*, Collect. Math. 64 (2013), 313–350.

[70] Y. Sawano and T. Shimomura, *Sobolev's inequality for Riesz potentials of functions in generalized Morrey spaces with variable exponent attaining the value 1 over non-doubling measure spaces*, J. Inequal. Appl. 2013 (2013), 19 pp.

[71] Y. Sawano and T. Shimomura, *Sobolev embeddings for Riesz potentials of functions in Musielak–Orlicz–Morrey spaces over non-doubling measure spaces*, Integral Transforms Spec. Funct. 25 (2014), 976–991.

[72] Y. Sawano and H. Tanaka, *The Fatou property of block spaces*, J. Math. Sci. Univ. Tokyo 22 (2015), 663–683.

[73] A. Seeger and T. Tao, *Sharp Lorentz space estimates for rough operators*, Math. Ann. 320 (2001), 381–415.

[74] Z. Shen, *Boundary value problems in Morrey spaces for elliptic systems on Lipschitz domains*, Amer. J. Math. 125 (2003), 1079–1115.

[75] J. Tao, Da. Yang and Do. Yang, *Boundedness and compactness characterizations of Cauchy integral commutators on Morrey spaces*, Math. Methods Appl. Sci. 42 (2019), 1631–1651.

[76] F. Wang, D. Yang and S. Yang, *Applications of Hardy spaces associated with ball quasi-Banach function spaces*, Results Math. 75 (2020), Paper No. 26, 58 pp.

[77] X. Yan, Z. He, D. Yang and W. Yuan, *Hardy spaces associated with ball quasi-Banach function spaces on spaces of homogeneous type: Characterizations of maximal functions, decompositions, and dual spaces*, Math. Nachr. 296 (2023), 3056–3116.

[78] W. Yuan, W. Sickel and D. Yang, *Morrey and Campanato Meet Besov, Lizorkin and Triebel*, Lecture Notes in Math. 2005, Springer-Verlag, Berlin, 2010.

[79] Y. Zhang, L. Huang, D. Yang and W. Yuan, *New ball Campanato-type function spaces and their applications*, J. Geom. Anal. 32 (2022), Paper No. 99, 42 pp.

[80] Y. Zhang, D. Yang, W. Yuan and S. Wang, *Weak Hardy-type spaces associated with ball quasi-Banach function spaces I: Decompositions with applications to boundedness of Calderon–Zygmund operators*, Sci. China Math. 64 (2021), 2007–2064.

[81] Y. Zhao, Y. Sawano, J. Tao, D. Yang and W. Yuan, *Bourgain–Morrey spaces mixed with structure of Besov spaces*, Proc. Steklov Inst. Math. 323 (2023), 244–295.

[82] C. Zhu, D. Yang and W. Yuan, *Brezis–Seeger–Van Schaftingen–Yung-type characterization of homogeneous ball Banach Sobolev spaces and its applications*, Commun. Contemp. Math. 26 (2024), Paper No. 2350041, 48 pp.

Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, The People's Republic of China

Email address: yiqunchen@mail.bnu.edu.cn

Department of Mathematics, University of Missouri, Columbia MO 65211, USA

Email address: grafakosl@missouri.edu

Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, The People's Republic of China

Email address: dcyang@bnu.du.cn

Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, The People's Republic of China

Email address: wenyuan@bnu.du.cn